Exponential Functionals of Markov Additive Processes

by

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Thesis

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Contents

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Declarations

This thesis is submitted to the University of Warwick in support of my application for the degree of Doctor of Philosophy. The work presented was carried out by the author, under the supervision of Larbi Alili (University of Warwick), and has not been submitted in any previous application for any degree.

Two pieces of literature have been produced from this work. A list of papers is provided below.

- L. Alili, D. Woodford: A comparison of European and Asian options under Markov additive processes. *Submitted to the SIAM Journal on Financial Mathematics* (2019), arXiv:1907.06596.
- L. Alili, D. Woodford: On finiteness and tails of perpetuities under a Lamperti-Kiu MAP. *In preparation* (2019), arXiv:1811.10286.

Abstract

This thesis is a study of the exponential functionals of Markov Additive Processes (MAPs) and their applications to the valuation of European and Asian options. Using the theory of exponential functionals of Lévy processes as a guide, the finiteness, (fractional complex) moments, and right tails, of the exponential functional of a MAP are considered for different classes of MAP. This includes MAPs which satisfy a Cramér type condition and MAPs of a strong subexponential type.

The theory of generalised Gamma functions is considered in the matrix setting, to solve a functional equation satisfied by the Mellin transform of the density of the exponential functional of a MAP. In certain cases, this theory is used to find the Mellin transform and obtain an asymptotic expansion of the right tails.

More generally, under a Cramér type condition, it is shown that the right tails are of a polynomial order. When the MAP is of strong subexponential type, and so the Cramér type condition fails, the right tail of the exponential functional is obtained at a log level.

The application of MAPs to the pricing of European and Asian options is also studied. It is shown that the Mellin transform and partial integro-differential equation methods for pricing European options can be used in an exponential MAP model. The link between the payoff of an Asian option and the exponential functional is then used to obtain a pricing method for Asian options. It is also shown how martingale properties of MAPs can be used to compare the prices of European and Asian options in certain cases.

Abbreviations and Notation

The following symbols are used throughout this thesis.

Sets of numbers

$\mathbb{R}$	Real numbers.
$\mathbb{R}^*$	Real numbers excluding zero.
$\mathbb{R}^+$	Strictly positive real numbers.
$\mathbb{N}$	Natural numbers excluding 0.
$\mathbb{N}_0$	Natural numbers including 0.
$\mathbb{Z}$	Integers.
$\mathbb{C}$	Complex numbers.
$\mathbb{C}^+$	Complex numbers with strictly positive real part.
$\mathbb{C}^-$	Complex number with strictly negative real part.
$\mathcal{S}_{a,b}$	For $a, b \in \mathbb{R}$, $\mathcal{S}_{a,b} \coloneqq \{s \in \mathbb{C} : a < \Re(s) < b\}$.
$\mathcal{S}_{a,b}^T$	For $a, b \in \mathbb{R}$ and $T > 0$, $\mathcal{S}_{a,b}^T \coloneqq \{s \in \mathcal{S}_{a,b} \,:\, \lvert\Im(s)\rvert > T\}$.

The notation $\mathbb{E}$ denotes expectation with respect to $\mathbb{P}$ and, if $\mathbb{E}$ has a subscript, then the expectation should be taken with respect to the probability measure with the same subscript.

Integral transforms

$\mathcal{F}$	Fourier transform.
$\mathcal{L}$	(Unilateral) Laplace transform.
$\mathcal{M}$	Mellin transform.

In general, an integral transform $\mathcal{T}$ applied to the function f and evaluated at x will be written as $\{\mathcal{T}f\}(x)$. The function f may be written in terms of a dummy variable. A subscript to the transform will sometimes be used to indicate which variable the transform should be taken with respect to.

Other Symbols

$\stackrel{\mathcal{L}}{=}$	Equality in law.
$:=$	Is defined by.
$x \wedge y$	Minimum of x and y.
$x \vee y$	Maximum of x and y.
$\Re(x)$	Real part of $x \in \mathbb{C}$.
$\Im(x)$	Imaginary part of $x \in \mathbb{C}$.

Limiting Behaviour and Bounds

For two functions $f, g : \mathbb{R} \to \mathbb{R}$, we write, for $a \in [-\infty, +\infty]$:

$f \sim g$ as $x \to a$	if $\lim_{x\to a} f(x)/g(x) = 1$;
$f = o(g)$ as $x \to a$	if $\lim_{x\to a} f(x)/g(x) = 0$;
$f = \mathcal{O}(g)$ as $x \to a$	if $\lim_{x\to .a} \lvert f(x)\rvert / \lvert g(x)\rvert < \infty$

For $x, y \in \mathbb{C}$, write $x = \omega(y)$ if $|x| \leq |y|$. Also, let $\Omega \in \mathbb{C}^{2\times 2}$ denote any matrix in which all the entries are less than 1.

Matrices

Let $\prod$ denote the ordered product of matrices, so that for $n \in \mathbb{N}$ and matrices $A_1, \ldots, A_n$,

$$\prod_{k=1}^{n} A_k := A_1 A_2 \ldots A_n \qquad \text{and} \qquad \prod_{k=n}^{1} A_k := A_n A_{n-1} \ldots A_1.$$

Unless otherwise specified, we will use the L^1-norm on $\mathbb{C}^2$ and the matrix norm induced by it on $\mathbb{C}^{2\times 2}$. That is, $\|A\| := \sup_{x\in\mathbb{C}^2\setminus\{0\}} \frac{\|Ax\|}{\|x\|}$ for any matrix $A \in \mathbb{C}^{2\times 2}$.

Common Abbreviations

ssMp	Self-similar Markov process.
MAP	Markov Additive Process.
i.i.d.	Independently and identically distributed.
PDE	Partial Differential Equation.
PIDE	Partial Integro-Differential Equation.
a.s.	Almost surely.

References and Cross-References

References to items of literature are denoted by square brackets, [ ], in the text and are detailed in full in the Bibliography. When cross-referencing an equation from within this document round bracktes, ( ), are used and cross-references to the sections, theorems and other numbered parts of this document are given explicity by their number and title.

Chapter 1

Introduction

1.1 Introduction

This thesis consists of a study of the *exponential functionals* of *Markov Additive Processes* (MAPs) and their applications to the valuation of *European* and *Asian options*.

A MAP can be thought of as a *Lévy process* with a Lévy triplet that varies depending on the value of a continuous time, finite state space Markov chain. More precisely, let E be a finite set and suppose $(\mathcal{F}_t)_{t\geq 0}$ is a filtration. Then, a pair of processes (J, ξ), taking values in $E\times\mathbb{R}$, is a Markov Additive Process (MAP) with respect to $(\mathcal{F}_t)_{t\geq 0}$ if, for any continuous bounded function $f : E \times \mathbb{R} \to \mathbb{R}^+$, $(z, y) \in E \times \mathbb{R}$ and $s, t \geq 0$, we have

$$\mathbb{E}_{z,y}[f(J_{t+s}, \xi_{t+s} - \xi_t)|\mathcal{F}_t] = \mathbb{E}_{J_t,0}[f(J_s, \xi_s)], \tag{1.1.1}$$

where $\mathbb{P}_{z,y}$ is the law of (J, ξ) started at (z, y) and $\mathbb{E}_{z,y}$ is the corresponding expectation. Of particular interest is the case that $E = \{+, -\}$ where, following [11], we will refer to the process $\{J_t \exp(\xi_t) \,:\, t \geq 0\}$ as a *Lamperti-Kiu* process.

A detailed account of MAPs is given in [4, Chapter XI], whilst a more general definition is given in [1, Section 3, pp 10, Definition 1]. The earliest definition of a MAP that I am aware of was given in [13]. Early works regarding MAPs were carried out in the sequence of papers [12].

For a MAP (J, ξ), the *standard exponential functional* is defined by

$$A_\infty := \int_0^\infty \exp(\xi_t)dt \tag{1.1.2}$$

and in the Lamperti-Kiu case the *signed exponential functional* is defined as

$$B_\infty := \int_0^\infty J_t \exp(\xi_t) dt.$$

The random variables A_∞ and B_∞ have been studied in a number of recent papers, such as [35] and [51]. Within this thesis their finiteness, tails, (fractional) moments and density are considered. As a guide to this study, use is made of the extensive literature on the exponential functional of Lévy processes, which is outlined within Chapter 2. Applications of MAPs to finance are then considered.

Whilst many of the results are stated for the Lamperti-Kiu case, where $|E| = 2$, they may be readily extended to any finite E. I believe that the case of an infinite E would require a different approach, however.

1.2 Applications to Finance

The use of stochastic processes to model financial assets and price derivatives written against them began with the well known *Black-Scholes* model [7], [40], which assumes asset prices are modelled by *geometric Brownian motion.* Since its publication, many different stochastic processes have been used to try to address the deficiencies of geometric Brownian motion. These deficiencies include the assumptions that: (i) prices are continuous in time; and (ii) the model parameters (the volatility) are constant over time. Lévy processes are a natural way to allow discontinuous paths whilst maintaining independent and identically distributed (i.i.d) increments (for example, see [52]). *Markov modulated processes* are commonly used to allow changes in the parameters over time (for example, see [41] and [19]).

Several authors have attempted to combine these two concepts through *Markov modulated jump diffusions* or *regime switching jump diffusions.* These models are a subset of MAPs. There remain gaps in the literature of their applications to finance, however.

Within this thesis the value of *European* and *Asian* options is considered under an exponential MAP model. This approach assumes the price process is modelled by $\{Y_t := \exp(\xi_t) : t \geq 0\}$ for some MAP (J, ξ). The process $\{J_t : t \geq 0\}$ can be thought of as describing the market regime. In particular, an exponential MAP model allows the asset price to follow any Lévy process between regime changes. It also allows the process to jump at the times that the regime switches.

A *European option* with *payoff* $H : \mathbb{R}^+ \to \mathbb{R}$ and *maturity* $T > 0$, pays the buyer $H(Y_T)$ at time T. The corresponding *Asian option*, with the same payoff function, H, and maturity, T, starting at time $t_0 < T$, pays the buyer $H(I_T)$ at time T, where

$$I_T := \frac{1}{T - t_0} \int_{t_0}^{T} Y_s ds. \tag{1.2.1}$$

The payout and hence the value of an Asian option is therefore path dependent. Under an exponential MAP model, the value of an Asian option is closely related to the exponential functional of a MAP.

1.3 Other Motivations

In addition to the applications to finance studied in this thesis, MAPs and their exponential functionals are also used in many other areas of probability. A brief outline of some of these applications is given below.

MAPs are related to *self-similar Markov process (ssMp)* through the *Lamperti transform*, first shown in [32] and further studied in [11]. Let $E = \{+, -\}$ and consider the MAP $\{(J_t, \xi_t) : t \geq 0\}$ from (1.1.1). For $\alpha \in (0, \infty)$, define a time transformation τ by

$$\tau(t) := \inf \left\{ u \geq 0 \ : \ \int_0^u \exp\left(\alpha \xi_s\right) ds \geq t \right\}.$$

Then, for all $x \in \mathbb{R}^* := \mathbb{R}\backslash\{0\}$, the process $X_t^{(x)} := J_{\tau(t|x|^{-\alpha})} x \exp(\xi_{\tau(t|x|^{-\alpha})})$ for $t < |x|^\alpha \int_0^\infty \exp\left(\alpha \xi_s\right) ds$ is a *self-similar Markov process* (ssMp) of index α, taking values in $\mathbb{R}^*$ and started at x. That is, X is a càdlàg Markov process such that, for all $c > 0$ and $x \in \mathbb{R}^*$, it satisfies the relation

$$\left\{ c X_{c^{-\alpha} t}^{(x)} \ : \ t \geq 0 \right\} \stackrel{\mathcal{L}}{=} \left\{ X_t^{(cx)} \ : \ t \geq 0 \right\}.$$

Moreover, this transformation is a bijection: any ssMp of index $\alpha > 0$ and taking values in $\mathbb{R}^*$ can be obtained from a MAP with state space $\{+, -\} \times \mathbb{R}$ in this way. This transformation and its inverse are known as the Lamperti transform and are well studied in the literature (see for instance [6], [11], [32], [37] and [38]). Then, the first hitting time of zero by $X^{(x)}$ is given by $\int_0^\infty \exp(\alpha \xi_s) ds$ in general, and the exponential functional A_∞ when $\alpha = 1$. This motivates a study of the exponential functional of a MAP in [35].

The application of MAPs and their exponential functionals to *multi-type self-similar fragmentation processes and trees* is considered in [51]. Important properties, including the finiteness and integer moments, of the exponential functional of non-increasing MAPs are given in [51, Section 1].

MAPs are also commonly studied in connection with *multi-server queues*, indeed, one of the standard references for MAPs, [4], is primarily a textbook on queuing. However, the exponential functionals have found less applications to this topic.

1.4 Summary

Chapter 2 consists of a literature review. MAPs and their exponential functionals are introduced and some background results are stated. Results from the literature on the exponential functionals of Lévy processes, which will guide the direction of this thesis, are also given.

In Chapter 3, some preliminary results which are not explicitly stated in the literature are proven. In Theorem 3.1.1, equivalent conditions for the integrability and uniform integrability of $\exp(\xi_t)$ are given. It is also shown that $\exp(\xi_t)$ can not be strictly locally integrable, nor can it be a strictly local martingale. Then, in Theorem 3.2.1 a full characterisation of the finiteness of the exponential functional is given. This includes the cases where the mean of ξ is undefined.

In Chapter 4, the *generalised Gamma functions* of [53] are extended to matrix valued functions in Theorem 4.1.1. For a matrix valued function $H : \mathbb{C} \to \mathbb{C}^{2\times 2}$, which satisfies certain assumptions, the corresponding generalised Gamma function $M : \mathbb{C} \to \mathbb{C}^{2\times 2}$ satisfies $M(0) = I$ and $M(s+1) = H(s)M(s)$. It is known that $\mathbb{E}[A_\infty^{s-1}]$, the Mellin transform of the density of A_∞, satisfies the similar relation

$$\left(\mathbb{E}_{0,\alpha}[A_\infty^{s+1}]\right)_{\alpha\in E} = -(s+1)(F(s+1))^{-1}\left(\mathbb{E}_{0,\alpha}[A_\infty^{s}]\right)_{\alpha\in E},$$

for $s > 0$ when $\mathbb{E}[A_\infty^{s+1}] < \infty$, where $F : \mathbb{C} \to \mathbb{C}^{2\times 2}$ is a matrix taking a similar role to the Laplace exponent of a Lévy process. Theorem 4.1.2 provides a method for verifying that a function satisfying this equation gives the Mellin transform of the density of A_∞. For two classes of MAPs (one of which is strictly decreasing and the other of which is spectrally positive with a non-zero diffusion component), these results are combined to express the Mellin transform as a generalised Gamma function in Section 4.2. Then, in Section 4.3, an explicit example is given, where the transform can be inverted to find the density of A_∞.

The right tails of the exponential functional A_∞ are studied in two significant cases in Chapter 5. In the first case, when a *Cramér type condition* is met, Theorem 5.1.1 shows that the right tails have polynomial decay, of an order given by Cramér's number. This result is achieved by considering a random affine equation solved by A_∞. Then, in the spectrally positive case considered in Section 4.2.2, an asymptotic expansion is given in Theorem 5.2.1. This is proven by considering the inverse Mellin tranform of the generalised Gamma function from Section 4.2.2.

The second case considered is when the MAP is of *strong subexponential type*: a subclass of MAPs with *heavy tails*. In this case, a different approach is needed since the positive moments of A_∞ are not finite. In Theorem 5.3.1, a careful study of the paths of (J, ξ) shows the right tails of $\log(A_\infty)$ are of the same order as the tails of ξ and hence are subexponential.

The valuation of European and Asian options under an exponential MAP model is studied in Chapter 6. Techniques for investigating the price of a European option using the Mellin transform and solving a partial integro-differential equation are applied to an exponential MAP model in Proposition 6.1.2 and Proposition 6.1.6, respectively. The connection between the value of an Asian option and the exponential functional of a MAP is described in Proposition 6.2.1. Then, by obtaining martingale results, a comparison is made between the prices of European and Asian options.

Chapter 2

Background and Literature Review

2.1 Lévy Processes

Lévy processes are closely related to MAPs and will be used extensively throughout this thesis. The following definition is from [36, pp 2, Chapter 1, Definition 1.1].

Definition 2.1.1 (Lévy Process)
A process $\{Z_t : t \geq 0\}$ is a *Lévy Process* with respect to a probability measure $\mathbb{P}$ if:

1. Z is *càdlàg* (right continuous with left limits) $\mathbb{P}$-almost surely;
2. $Z_t - Z_s$ is independent of $\{Z_u : u \leq s\}$ for all $t > s > 0$;
3. $Z_t - Z_s$ has the same distribution as Z_{t-s}.

An important result in the study of Lévy processes is the *Lévy-Khintchine* formula for the *Laplace exponent*. The *(positive) Laplace exponent* of Z is defined to be

$$\psi(x) \coloneqq \log(\mathbb{E}\left[\exp\left(xZ_1\right)\right]),$$

for all $x \in \mathbb{C}$ such that $\mathbb{E}\left[\exp\left(xZ_1\right)\right] < \infty$. The positive Laplace exponent is used here, rather than the more common *negative Laplace exponent*, since it fits more naturally with the conventions used for the matrix exponent of a MAP (for example in [4, Chapter 11]). Notice that $\psi(x)$ is always well defined for $x \in i\mathbb{R}$ and is well defined for all $x \in \mathbb{R}^+$ if Z is *spectrally negative* (has no positive jumps). The Lévy-Khintchine formula then states that there exists a unique triplet (a, σ, π) such that $a, \sigma \in \mathbb{R}$ and π is a measure concentrated on $\mathbb{R}^*$, which satisfies $\int_{\mathbb{R}} \min(1, u^2)\pi(du) < \infty$ and

$$\psi(x) = ax + \frac{\sigma^2}{2}x^2 + \int_{\mathbb{R}} \left(e^{xu} - 1 - xu\mathbb{1}_{\{|x|<1\}}\right) \pi(du),$$

for all $x \in \mathbb{C}$ such that $\psi(x)$ is well defined (for example, see [36, pp 5, Chapter 1, Theorem 1.6]). The measure π is referred to as the *Lévy measure*. The triplet (a, σ, π) fully describes the Lévy process and is referred to as the *characteristic triplet* or *Lévy triplet*. The Lévy-Khintchine formula is often stated in terms of the *characteristic exponent*, $\phi(\theta) := -\log\left(\mathbb{E}\left[\exp\left(i\theta Z_1\right)\right]\right) = -\psi(i\theta)$, in which case, for all $\theta \in \mathbb{R}$,

$$\phi(\theta) = -ai\theta + \frac{\sigma^2}{2}\theta^2 + \int_{\mathbb{R}} \left(1 - e^{i\theta u} + i\theta u\mathbb{1}_{\{|u|<1\}}\right) \pi(du).$$

A *subordinator* is a non-decreasing Lévy process, whilst a *negative subordinator* is a non-increasing Lévy process. A *spectrally positive* Lévy process is any Lévy process such that $\pi((-\infty, 0)) = 0$ and a *spectrally negative* Lévy process has $\pi((0, \infty)) = 0$. A *spectrally one-sided* Lévy process is a process which is either spectrally positive or spectrally negative. It is necessary that any subordinator is spectrally positive.

Another important class of Lévy processes in the study of exponential functionals are those which satisfy *Cramér's condition*. This condition is satisfied if there exists $\kappa > 0$ such that $\psi(\kappa) = 0$. Then, κ is referred to as *Cramér's number*.

2.2 Exponential Functional of Lévy Processes

A comprehensive survey of the exponential functional of a Lévy process is given in [6]. Suppose $\{Z_t : t \geq 0\}$ is a Lévy process with positive Laplace exponent ψ. Then, in [6, pp 192, Theorem 1], conditions equivalent to the finiteness of the exponential functional $I_\infty := \int_0^\infty \exp(-Z_t)dt$ are given and include $\lim_{t\to\infty} t^{-1}Z_t > 0$ almost surely.

Using the Markov property, it is observed in [6] that I_∞ satisfies the random affine equation,

$$I_\infty \overset{\mathcal{L}}{=} \int_0^T e^{-Z_t}dt + e^{-Z_T}\hat{I}_\infty,$$

where $\hat{I}_\infty$ denotes an independent and identically distributed copy of I_∞. This type of equation has been studied extensively, for example, see [31] and [27]. Using these results, it is shown in [6] that if there exists a $\kappa \in (0,\infty)$ such that $\psi(-\kappa)=0$, $\mathbb{E}[Z_t \exp(-\kappa Z_t)]<\infty$, $\mathbb{E}\left[\exp(-\kappa Z_t)\log^+ \exp(-Z_t)\right]<\infty$ and Z_1 has a non-lattice distribution, then

$$\mathbb{P}(I_\infty > x) \sim cx^{-\kappa}, \tag{2.2.1}$$

for some constant $c>0$, as $x\to\infty$. Recall from Section 2.1 that such a $-\kappa$ is Cramér's number. Obtaining the value of the constant c is a challenging problem.

Under the assumption that the Lévy process Z is a subordinator with Laplace exponent ψ, it is established in the proof of [6, pp 194, Theorem 2] that I_∞ satisfies the recurrence relation,

$$\mathbb{E}[I_\infty^s] = \frac{-s}{\psi(-s)}\mathbb{E}[I_\infty^{s-1}], \tag{2.2.2}$$

for $s>0$. In [39], this is extended to any Lévy process and holds for all $s>0$ such that $\psi(-s)<0$.

From these relations, it is straight forward to obtain the positive and negative entire moments of I_∞ as

$$\mathbb{E}[I_\infty^m] = \frac{(-1)^m m!}{\psi(-1)\dots\psi(-m)} \qquad \text{and} \qquad \mathbb{E}[I_\infty^{-m}] = \psi'(0)\frac{\psi(-1)\dots\psi(1-m)}{(-1)^m(m-1)!}, \tag{2.2.3}$$

respectively, for $m\in\mathbb{N}$, when they exist. In the case of a subordinator, [6, pp 194, Theorem 2] gives all of the positive moments and [6, pp 197, Theorem 3] gives the negative moments, provided the Lévy process has exponential moments of all negative orders.

The problem of finding $\mathbb{E}[I_\infty^s]$ for non-integer values of s has been well studied in the literature. Since $\mathbb{E}[I_\infty^{s-1}]$ corresponds to the *Mellin transform* of the density of I_∞, if it is found within some vertical strip of $\mathbb{C}$, then the density can be obtained via the *inverse Mellin transform*.

To find the Mellin transform of the density of I_∞ requires uniquely solving the functional equation (2.2.2), such that the solution interpolates (2.2.3), within a suitable space of functions. The general problem of finding $f:\mathbb{R}\to\mathbb{R}$ such that for all $x\in\mathbb{R}^+$,

$$f(x+1)=g(x)f(x) \qquad \text{and} \qquad f(1)=1, \tag{2.2.4}$$

for some eventually log-concave function g, is solved in the space of log-convex functions in [53]. The unique solution is given by the *generalised Gamma function*:

$$\Gamma_g(x) := \frac{e^{-\gamma_g x}}{g(x)}\prod_{n=1}^{\infty}\frac{g(n)e^{((g'_-(x)+g'_+(x))/2g(n))x}}{g(n+x)}, \tag{2.2.5}$$

where $g'_\pm(x)$ are the left and right derivatives of g at x and γ_g is a constant depending on g. In the case that $g(x) \coloneqq x$, the function Γ_g corresponds to the *Weierstrass definition* of the Gamma function (see Appendix A.1).

If Z is a subordinator, then [2, pp 6, Section 1, Theorem 2] gives, for $s > -1$,

$$\mathbb{E}\left[I_\infty^s\right] = \frac{\Gamma(s+1)}{\Gamma_{-\psi}(s+1)},$$

whilst a result for a subclass of these process is given in [39, pp 5, Section 2.2, Proposition 2.2]. If Z is spectrally negative with negative drift and a non-zero diffusion component, then [39, pp 7, Section 2, Proposition 2.3] gives the result

$$\mathbb{E}[I_\infty^s] = \mathbb{E}[Z_1]\frac{\sigma^2}{2}e^{\gamma(s+1)}\prod_{k=1}^{\infty} e^{-(s+1)/k}\frac{\psi(k)}{\psi(k-s-1)}\frac{k-s-1}{k},$$

for all $s \in (-\infty, -\kappa) \setminus \mathbb{N}_0$. In the case that Z is a *hypergeometric Lévy process*, [34, pp 8, Section 3, Theorem 2] provides an expression for the Mellin transform in terms of the *double Gamma* function.

It is also proven in [34, pp 10, Section 3, Proposition 2] that if Z is any Lévy process satisfying Cramér's condition and $f : \mathbb{R} \to \mathbb{R}$ is a function such that: *(i)* f satisfies the functional equation (2.2.2) and $f(1) = 1$; *(ii)* f is analytic and zero-free in the strip $(0, 1+\kappa)$; and *(iii)* $|f(s)|^{-1} = o(\exp(2\pi|\Im(s)|))$ as $\Im(s) \to \infty$ for all $\Re(s) \in (0, 1+\kappa)$; then $f(s)$ is the Mellin transform of the density of I_∞. This provides a method of verifying whether a given solution to the functional equation is the Mellin transform of the density of I_∞.

A more general approach is taken in [44]. Here the relation (2.2.2) is solved for all Lévy-Khintchine representations by splitting them into their *Wiener-Hopf* factors and solving for each factor separately. Thus, if $-\psi(-s) = -\phi_+(-s)\phi_-(s)$ is the Wiener-Hopf factorisation of a Lévy-Khintchine representation ψ, then (2.2.2) is solved by

$$M_\psi(s) = \frac{\Gamma(s)}{\Gamma_{\phi_+}(s)}\Gamma_{\phi_-}(1-s).$$

If Z is a Lévy process corresponding to ψ and I_∞ is its exponential functional, then

$$\mathbb{E}[I_\infty^{s-1}] = \phi_-(0)M_\psi(s).$$

From these representations of $\mathbb{E}[I_\infty^s]$, many other properties of the density of I_∞ have been deduced. Of particular interest are examples where the inverse Mellin transform can be computed explicitly to obtain the density of the exponential functional. Also of interest are examples where the tails of the density have been obtained.

Under Cramér's condition, with Cramér's number κ, [39, pp 9, Section 3, Theorem 3.1] and [44, pp 11, Section 2, Theorem 2.11(2)] show that the exponential functional has right tails given by

$$\mathbb{P}(I_\infty > x) \sim \frac{\mathbb{E}[I_\infty^{\kappa-1}]}{-\psi'(\kappa)} x^{-\kappa},$$

as $x \to \infty$, where $\mathbb{E}[I_\infty^{\kappa-1}]$ can be evaluated using generalised Gamma functions. This then gives a way of computing the constant c in (2.2.1).

In the hypergeometric case (with parameter set $(1-\alpha(1-p), \alpha p, 1-\alpha(1-p), \alpha(1-p)))$, [34, pp 14, Theorem 3] establishes the tail of the density of I_∞, as $x \to +\infty$ and $x \to 0^+$, in the form of an infinite power series for all $\alpha \notin \mathbb{Q}$. In [34, pp 15, Theorem 4], an infinite power series is established for the density of I_∞ for α in a subset of the irrational numbers.

More generally, using the generalised Gamma functions associated with the Wiener-Hopf factorisation, [44, pp 11, Section 2, Theorem 2.11(1)] determines the right tails at a *log* level for all Lévy processes, stating

$$\lim_{x\to\infty} \frac{\log \mathbb{P}(I_\infty > x)}{\log(x)} = -\kappa^*,$$

where $\kappa^* = -\kappa$ if Cramér's condition is met and $\kappa^* := \sup\{u > 0 : \psi \text{ is analytic on } (-\infty, u)\}$ otherwise.

Each of these expressions for the Mellin transform, density and tails of I_∞ requires the existence of ψ on a subset of $\mathbb{R}^+$ and therefore the existence of exponential moments of Z. In [39, Section 4], some cases where this fails are considered. Under the assumption that $G(x) := \min\left(1, \int_x^\infty \mathbb{P}(-Z_1 > u)du\right)$ is *subexponential* as a function of x and $\mathbb{E}[Z_1] > 0$, it is shown that, on a log level, I_∞ has tails given by

$$\mathbb{P}(\log(I_\infty) > x) \sim \frac{G(x)}{\mathbb{E}[Z_1]}.$$

2.3 Markov Additive Processes

Markov Additive Processes (MAPs) are the stochastic processes that underly the results of this thesis. Recall (1.1.1) from the introduction:

Definition 2.3.1 (Markov Additive Process)
Let E be a finite set and suppose $(\mathcal{F}_t)_{t\geq 0}$ is a filtration. A pair of processes (J, ξ), taking

values in $E \times \mathbb{R}$, is a *Markov Additive Process* (MAP) with respect to $(\mathcal{F}_t)_{t\geq 0}$ if, for any continuous bounded function $f : E \times \mathbb{R} \to \mathbb{R}^+$, $(z, y) \in E \times \mathbb{R}$ and $s, t \geq 0$, we have

$$\mathbb{E}_{z,y}[f(J_{t+s}, \xi_{t+s} - \xi_t)|\mathcal{F}_t] = \mathbb{E}_{J_t,0}[f(J_s, \xi_s)], \tag{2.3.1}$$

where $\mathbb{P}_{z,y}$ is the law of (J, ξ) started at (z, y) and $\mathbb{E}_{z,y}$ is the corresponding expectation.

This definition is based upon the one given in [4, Chapter X1, Section 2], which is used as our main reference. A more general definition, with less restrictions on the set E, can be found in [13]. This was one of the earliest definitions of a MAP.

Definition 2.3.2 (Lamperti-Kiu Process)
Following the convention of [11], in the case $E := \{+, -\}$ we will refer to the process $\{Y_t := J_t \exp(\xi_t) \, : \, t \geq 0\}$ as a *Lamperti-Kiu process*.

When considering such a MAP, for all $\alpha \in E$ we let $\mathbb{P}_\alpha$ denote the probability measure given that $(J_0, \xi_0) = (\alpha, 0)$ and let $\mathbf{P} := (\mathbb{P}_\beta)_{\beta \in E}$ denote an $|E|$ dimensional vector of probability measures. Then, let $\mathbb{E}$ and $\mathbf{E}$ denote the corresponding expectations. If $(\alpha, z) \in E \times \mathbb{R}$, then we let $\mathbb{P}_{\alpha,z}$ denote the probability measure given $(J_0, \xi_0) = (\alpha, z)$ and $\mathbb{E}_{\alpha,z}$ denote the corresponding expectation.

Much of the literature of MAPs is built upon two important results. The first is the existence of a decomposition of a MAP into simpler parts. The second is the existence of a matrix which takes the role of the (positive) Laplace exponent of a Lévy process, known as the *matrix exponent* of the MAP. Both of these results emphasize the connection between Lévy processes and MAPs.

2.3.1 Decomposition of MAPs

A *decomposition* of MAPs is given in general in [4, pp 309, Part C, Chapter XI, Section 2] and in the case $|E| = 2$ in [11, pp 2502, Theorem 6(i)] and [17, pp 3, Section 1.2]. This states that the MAP (J, ξ) can be decomposed as follows.

For each $\alpha \in E$, there exists a Lévy process, $\xi^{(\alpha)}$, with characteristic triplet $(a_\alpha, \sigma_\alpha, \mu_\alpha)$. For each $\alpha, \beta \in E$, there exists an exponentially distributed random variable, $\zeta_{\alpha,\beta}$, with rate $q_{\alpha,\beta} \geq 0$ and a random variable, $U_{\alpha,\beta}$, taking values in $\mathbb{R}$, with measure $\nu_{\alpha,\beta}$. There are sequences $(\xi^{(\alpha,k)})_{k\in\mathbb{N}}, (\zeta_{\alpha,\beta,k})_{k\in\mathbb{N}}$ and $(U_{\alpha,\beta,k})_{k\in\mathbb{N}}$, which are i.i.d. copies of $\xi^{(\alpha)}$, $\zeta_{\alpha,\beta}$ and $U_{\alpha,\beta}$, respectively, such that the sequences are also independent of each other. To simplify notation, also define $q_\alpha := -q_{\alpha,\alpha} := \sum_{\gamma \in E\setminus\{\gamma\}} q_{\alpha,\gamma}$.

Let $\{T_n\}_{n\in\mathbb{N}_0}$ denote the times at which the process J changes value, with the convention that $T_0 = 0$ and let $\zeta_k := \zeta_{J_{T_{k-1}},J_{T_k},k} = T_k - T_{k-1}$ for each $k \in \mathbb{N}$. Then, these objects are such that the process J is a continuous time Markov chain, with transition rate matrix $(q_{\alpha,\beta})_{\alpha,\beta\in E}$, and $T_n = \sum_{k=0}^{n-1} \zeta_k$. Then, setting

$$N_t := \max_{n\in\mathbb{N}_0} \{T_n \leq t\} \qquad \text{and} \qquad \pi_t := t - T_{N_t},$$

the process $\{\xi_t : t \geq 0\}$ is given by

$$\xi_t = \xi^{(N_t)}_{\pi_t} + \sum_{k=0}^{N_t-1} \left(\xi^{(k)}_{\zeta_k-} + U_k\right), \qquad t \geq 0, \tag{2.3.2}$$

where, for each $k \in \mathbb{N}$, we let $\xi^{(k)} := \xi^{(J_{T_{k-1}},k)}$ and $U_k := U_{J_{T_{k-1}},J_{T_k}}$.

Moreover, any process of this form is a MAP. We refer to this as the decomposition of (J, ξ). We will assume throughout that J is irreducible to exclude cases which degenerate into Lévy processes.

To simplify notation, in the case of a *Lamperti-Kiu* process, where $E = \{+,-\}$, we will replace the subscripts $(+,-)$ and $(-,+)$ of the components of the decomposition with $+$ and $-$, respectively. We will then refer to this as the *Lamperti-Kiu decomposition*.

2.3.2 Matrix Exponent and Cramér's Condition

Another important property of MAPs is the existence of a matrix which takes the role of the positive Laplace exponent of a Lévy process. This is referred to as the *matrix exponent* of the MAP. Suppose $z \in \mathbb{C}$ such that $\mathbb{E}[\exp(z\xi_t)] < \infty$. Then, from [4, pp 311, XI.2b], it is known that there exists a matrix $F(z) \in \mathbb{C}^{|E|\times|E|}$ such that, for all $\alpha, \beta \in E$ and $t \geq 0$,

$$\mathbb{E}\left[e^{z\xi_t};\, J_t = \beta \;\middle|\; J_0 = \alpha\right] = \left(e^{tF(z)}\right)_{\alpha,\beta}. \tag{2.3.3}$$

Moreover, for each $\alpha, \beta \in E$,

$$(F(z))_{\alpha,\beta} = \begin{cases} \psi_\alpha(z) - q_\alpha, & \text{if } \alpha = \beta; \\ q_{\alpha,\beta} G_{\alpha,\beta}(z), & \text{if } \alpha \neq \beta; \end{cases} \tag{2.3.4}$$

where, $\psi_\beta(z) := \log \mathbb{E}[\exp(z\xi^{(\beta)}_1)]$ and $G_{\alpha,\beta}(z) = \mathbb{E}[\exp(zU_{\alpha,\beta})]$. A detailed discussion of the matrix exponent F can be found in [4, pp 311, XI.2b].

Let $\lambda(z)$ denote the eigenvalue of $F(z)$ with largest real part (often referred to as the *principal eigenvalue*). Using *Perron-Frobenius* theory, it is shown in [35, pp 8, Proposition

3.2] that such an eigenvalue is guaranteed to be simple, real and continuous as a function of z, within the regions that it is well defined. From [35, Section 3, pp 10, Proposition 3.4], we also know that $\lambda(z)$ is convex. Then, provided F exists in a neighbourhood of zero, from [4, pp 313, Corollary 2.9] it is known that $\lambda'(0) \in [-\infty, \infty]$ and $\lambda(0) = 0$.

A *Cramér type condition* for MAPs is given in [35, pp 9, Section 3, Assumption 3.5] as: *there exists $\kappa > 0$ such that $F(s)$ exists for all $s \in (0, \kappa)$ and $\lambda(\kappa) = 0$. Then, κ is referred to as Cramér's number*.

If $F(s)$ is well defined for all $s \in [0, \infty)$, then the above properties of λ mean it can be characterised by one three possible cases:

1. *Cramér's case*: If Cramér's condition is met, then $\lambda'(0) < 0$, $\lambda(s) < 0$ for all $s \in (0, \kappa)$ and $\lambda(s)$ is positive and increasing on (κ, ∞);

2. *Positive case*: $\lambda'(0) > 0$, $\lambda(s) > 0$ for all $s > 0$, $\lim_{s\to\infty} \lambda(s) = \infty$ and so Cramér's condition is not met;

3. *Negative case*: $\lambda'(0) < 0$, $\lambda(s) < 0$ for all $s > 0$, $\lim_{s\to\infty} \lambda(s) = -\infty$ and so Cramér's condition is not met.

From this characterisation, it is possible to define an *extended Cramér's number*, $\hat{\kappa}$, such that: in Cramér's case $\hat{\kappa} = \kappa$; in the positive case $\hat{\kappa} = 0$; and in the negative case $\hat{\kappa} = \infty$. The extended Cramér's number then satisfies the property $\lambda(s) < 0$ for all $s \in (0, \hat{\kappa})$ and $\lambda(s) > 0$ for all $s \in (\hat{\kappa}, \infty)$.

2.4 Exponential Functional of MAPs

The main object of study in this thesis is the *exponential functional* of a MAP. For the MAP (J, ξ), this is given by

$$A_\infty := \int_0^\infty e^{\xi_t} dt$$

and in the *Lamperti-Kiu* case the *signed exponential functional* is defined to be

$$B_\infty := \int_0^\infty J_t e^{\xi_t} dt.$$

Moreover, for $T \in (0, \infty)$, we also define $A_T := \int_0^T e^{\xi_t} dt$ and, in the *Lamperti-Kiu* case, $B_T := \int_0^T J_t e^{\xi_t} dt$.

2.4.1 Moment Recurrence Relation

Under the assumption that the Cramér type condition is satisfied with Cramér's number $\kappa \in (0,1)$, the recurrence relation (2.2.2) is generalised to MAPs in [35, Proposition 3.6] by the relation

$$\mathbf{E}[A_\infty^s] = -s(F(s))^{-1}\mathbf{E}[A_\infty^{s-1}], \qquad \Re(s) \in (0,\kappa), \tag{2.4.1}$$

where $\mathbf{E}[A^s] := (\mathbb{E}_\alpha[A^s])_{\alpha\in E} \in \mathbb{C}^{|E|}$ is a vector as defined in Section 2.3. In [51, pp 9, Section 1.4, Lemma 1.9] and [51, Section 1.4, Lemma 1.11], this relation is shown to hold for non-increasing MAPs, for all $s \geq 0$ and for negative $s \in \mathbb{R}$ when $F(s)$ is well defined. In fact, it can be shown that this relation holds for all $s \in (\rho+1, \hat{\kappa})$ where $\rho := \inf\{z \in \mathbb{R} : \mathbb{E}[e^{z\xi_1}] < \infty\}$ and $\hat{\kappa}$ is the extended Cramér's number of Section 2.3.2.

This recurrence relation can be used to obtain the (positive and negative) entire moments of A_∞. In the case of a non-increasing MAP, it is shown in [51, Section 1.4, Proposition 1.8], that for any $n \in \mathbb{N}$,

$$\mathbf{E}\left[A_\infty^n\right] = \prod_{k=0}^{n-1}\left(-\frac{F(n-k)}{n-k}\right)^{-1}\mathbf{e}, \tag{2.4.2}$$

where $\mathbf{e} = (1,1)^T$, whilst in [51, Section 1.4, Proposition 1.10], it is shown that

$$\mathbf{E}\left[A_\infty^{-n}\right] = \prod_{k=n-1}^{0}\left(\frac{F(-k)}{k}\right)\mathbf{m}, \tag{2.4.3}$$

where $\mathbf{m} := \mathbb{E}[A_\infty^{-1}] = F'(0)\mathbf{e} + F(0)\mathbf{E}[\log A_\infty]$. Since the recurrence relation holds more generally, these results are easily extended to any MAP such that $\mathbf{E}[A_\infty^{\pm n}] < \infty$.

In [35, pp 13, Section 3.4, Theorem 3.9], an analytic expression is found for the Mellin transform of the density of A_∞ for MAPs corresponding to the Lamperti transform of $\mathbb{R}^*$ valued α-stable processes with two-sided jumps, that is, a stable process with $\alpha \in (1,2)$ and $E = \{+1,-1\}$.

2.5 Finance Applications

2.5.1 Option Pricing Under Exponential Lévy Models

Pricing of derivatives under exponential Lévy models is a well studied topic, for example, [52] can be taken as a reference. The Lévy-Khintchine representation of the Fourier transform of the marginal density of a Lévy process enables Fourier transform methods to be used to price European options, for instance see [10].

For example, consider a *European call option* where the buyer has the right, but not the obligation, to buy an asset for the *strike* price, $k > 0$, at the *maturity* time, $T > 0$. Suppose that the price process of the underlying asset is given by a Lévy process $\{Z_t : t \geq 0\}$. Let the positive Laplace exponent of the Lévy process be denoted by $\psi(x) := \log\left(\mathbb{E}\left[e^{xZ_1}\right]\right)$. Then, [10] shows that the Fourier transform, with respect to k, of the price of the call option, C, is given by

$$\{\mathcal{F}C\}(s) = \frac{e^{-rT}e^{\psi(s-(\alpha+1)i)}}{\alpha^2 + \alpha - s^2 + i(2\alpha+1)s},$$

where $\alpha > 0$ is some constant that we are free to choose. This method allows fast numerical computation of option prices using the *Fast Fourier Transform*. It also allows for option prices for all strikes $k > 0$ to be computed simultaneously.

In analogue to the *Black-Scholes PDE* [7], it is possible to derive a *partial integro-differential equation* (PIDE) for the price of a European option under an exponential Lévy model. For example, under some conditions on the density of Z_1, [15, pp 304, Section 2.1, Proposition 2] states that the price, $C(t, z)$, of a *European option*, with *payoff function* $H : \mathbb{R}^+ \to \mathbb{R}$ and maturity T, at time t when the asset price is $Z_t = z$, satisfies the PDE:

$$\begin{aligned} 0 = \frac{\partial C}{\partial t}(t,z) + rz\frac{\partial C}{\partial z}(t,z) + \frac{\sigma^2}{2}z^2\frac{\partial^2 C}{\partial z^2}(t,z) - rC(t,z)& \\ + \int_{\mathbb{R}} \left(C(t, ze^x) - C(t,z) - z(e^x - 1)\frac{\partial C}{\partial z}(t,z)\right) \mu(dx)&, \end{aligned}$$

on $[0, T) \times \mathbb{R}^+$, with boundary condition

$$C(T, z) = H(z).$$

2.5.2 Asian Options

The pricing of *Asian call options* under *geometric Brownian motion* is considered in [26]. It is shown that if the rate of interest is sufficiently high compared to the expected return of the stock, then an Asian call option is more expensive than the corresponding European option, for strikes in a neighbourhood of zero. This is in contrast to one of the commonly cited reasons for trading in Asian options.

In [26], it is shown that the Laplace transform of the value of an Asian call option with respect to time is given by the expected value of the exponential functional of a Brownian motion. In particular, if under the risk neutral measure an asset price is given by $\{\exp(B_t) : t \geq 0\}$

and e_q is an independent exponential random variable for each $q > 0$, then

$$\mathcal{L}\{C_A(T;k)\}(q) = \mathbb{E}\left[\left(\int_0^{e_q} \exp(B_t)dt - k\right)^+\right],$$

where $C_A(T;k)$ denotes the price of an Asian call option with maturity T and strike k. The Laplace transform is being taken with respect to T.

These are both methods that can be extended to option pricing under different models.

In the case of *meromorphic Lévy processes*, the pricing of Asian options is considered in [28]. This work is particularly relevant as they consider analytical approaches using Mellin/Laplace transforms of the exponential functional of the Lévy process.

2.5.3 Markov Modulated Jump Diffusions

Several authors have considered option pricing models that include jumps and have parameters specified by a continuous time Markov chain. These are often referred to as *Markov modulated jump diffusion* or *regime switching jump diffusion* models and are a subset of the class of exponential MAP models. Below is an outline of some of the more prominent examples in the literature.

One of the first such models was proposed in [41], where a model with geometric Brownian motion is considered, with the parameters of the Brownian motion depending upon a continuous time finite state space Markov chain and with additional jumps in the price process when the Markov chain changes value. However, this model does not include the presence of jumps when the model is in each regime. Similar models are considered in [47]. In the case that there are no jumps coinciding with regime changes, *American options* are considered in [9].

Elliot and Siu have also considered *Markov modulated jump diffusion's* in [19], [49] and [20], where the Lévy measure and drift is determined by a Markov chain. These papers view the system as a *hidden Markov model* and use *filtering techniques* to address the issue that the state of the Markov chain is unobservable. A generalisation of the *Esscher transform* is then applied to the filtered system, to obtain an equivalent martingale measure under which a PIDE can be derived to value European options. However, these models don't allow for a jump in price caused by a change in the regime and also consider the diffusion coefficient to be constant. Similar models are considered in [16] and [46].

In [8] a Markov modulated jump diffusion is suggested as a model for FX spot rates, where the states of the Markov chain correspond to the sovereign rating of the relevant countries or regions. This paper focuses on the case the jumps are *log-normally* distributed and doesn't consider jumps in the FX rate when the regime changes state. The authors are able to use a generalised Esscher transform to value European options under this model.

General exponential MAP models are considered in [42], in the context of the *portfolio selection problem*. By enlarging the market to include additional securities, they are able to find a *complete market* and give an *equivalent martingale measure*. These results form the basis of the pricing of derivatives under an exponential MAP model.

Chapter 3

Integrability of MAPs

3.1 Integrability

Let (J, ξ) be a MAP on $E \times \mathbb{R}$ for some finite set E and let $Y_t \coloneqq \exp(\xi_t)$, for all $t \geq 0$. We are often interested in when the expectation of Y exists, for example in Sections 6.1 and 6.3. Equivalent conditions for the existence of this expectation are given in the following theorem.

Theorem 3.1.1 (Integrability of an Exponential MAP)
Suppose (J, ξ) is a MAP on $E \times \mathbb{R}$, with decomposition (2.3.2) and matrix exponent F, and that $Y_t \coloneqq \exp(\xi_t)$, for all $t \geq 0$. Then, for all $p > 0$, the following are equivalent:

1. $\mathbb{E}[Y_t^p] < \infty$ *for all* $t \geq 0$;
2. *There exists some* $T > 0$ *such that* $\mathbb{E}[Y_T^p] < \infty$;
3. $\{Y_t^p : t \geq 0\}$ *is locally integrable;*
4. $\mathbb{E}\left[\exp\left(p\xi_1^{(\alpha)}\right)\right] < \infty$ *and* $\mathbb{E}\left[\exp\left(pU_{\alpha,\beta}\right)\right] < \infty$, *for all* $\alpha, \beta \in E$;
5. μ_α *and* $\nu_{\alpha,\beta}$ *have p-exponential moments for all* $\alpha, \beta \in E$;
6. $F(p)$ *exists.*

Moreover, if Cramér's number, κ, exists then $\{Y_t^p : t \geq 0\}$ is uniformly integrable, if and only if, $\kappa > p$.

This theorem is of particular interest as it relates the integrability of Y with properties of the components of its decomposition from Section 2.3.1. It also shows that existence of $F(p)$ is necessary and sufficient for integrability of Y^p, without conditioning on the states of J. Statement (3) extends the corresponding result of Lévy processes (see [45, Exercise 29, pp 49]) and we will also see that there are no strictly local martingales in Lemma 3.1.4.

Remark 3.1.1

Notice that $Y_t^p = \exp(p\xi_t)$ and also that $(J, p\xi)$ is a MAP. The Lévy processes in the decomposition (2.3.2) of $(J, p\xi)$ are given by $(p\xi^{(\alpha)})_{\alpha\in E}$ and the jumps induced by changes of J are $(pU_{\alpha,\beta})_{\alpha,\beta\in E}$. The matrix exponent of $(J, p\xi)$ is $F(pz)$ hence, if κ is Cramér's number for (J,ξ), then κ/p is Cramér's number for $(J, p\xi)$. Therefore, we need only consider the case $p = 1$.

The proof of Theorem 3.1.1 makes use of the following lemmas.

Lemma 3.1.1

There exists a Poisson process, $\{\eta_t : t \geq 0\}$, of rate $\lambda := \max_{\alpha\in E} q_\alpha$, such that $N_t \leq \eta_t$, for all $t \geq 0$.

Proof

For each $i \in \mathbb{N}_0$, since $\zeta_i \sim \mathrm{Exp}(q_i)$, there is a random variable X_i, uniformly distributed on $[0, 1]$, such that $\zeta_i = -\log(X_i)/q_i$. For each $n \in \mathbb{N}_0$, we have the inequality

$$T_n := \sum_{i=0}^{n-1} \zeta_i = \sum_{i=0}^{n-1} \frac{-\log(X_i)}{q_i} \geq \sum_{i=0}^{n-1} \frac{-\log(X_i)}{\lambda} = \sum_{i=0}^{n-1} \hat{\zeta}_i =: \hat{T}_n,$$

where $\{\hat{\zeta}_i\}_{i\in\mathbb{N}_0}$ is an i.i.d. sequence of exponential random variables of rate λ. Then, $\{\eta_t := \mathrm{argmax}_{n\in\mathbb{N}}\{T_n < t\} : t \geq 0\}$, is Poisson process of rate λ, such that $\eta_t \geq N_t$, for all $t \geq 0$. $\square$

The following lemma establishes the equivalence of statements (1) and (4) of Theorem 3.1.1.

Lemma 3.1.2

Y_t is integrable for all $t \geq 0$, if and only if, $\mathbb{E}\left[\exp\left(\xi_1^{(\alpha)}\right)\right] < \infty$ and $\mathbb{E}\left[\exp\left(U_{\alpha,\beta}\right)\right] < \infty$, for all $\alpha, \beta \in E$.

Proof

First, we suppose that the exponential moments of $\xi_1^{(\alpha)}$ and $U_{\alpha,\beta}$ exist for all $\alpha, \beta \in E$ and show that Y_t is integrable for all $t \geq 0$. Let $\hat{\xi} := \xi^{(\hat{\alpha})}$, where $\hat{\alpha} := \operatorname{argmax}_{\alpha \in E} \left\{ \mathbb{E}\left[\exp\left(\xi_1^{(\alpha)}\right)\right]\right\}$. Moreover, let $\left(\hat{\xi}^{(k)}\right)_{k\in\mathbb{N}}$ be a sequence of i.i.d. copies of $\hat{\xi}$, which are also independent of $\xi^{(\alpha,k)}$, for all $k \in \mathbb{N}_0$ and $\alpha \in E$. Then, by the independence structure of the decomposition (2.3.2), for all $k \in \mathbb{N}_0$ and $t \geq 0$,

$$\mathbb{E}\left[\exp\left(\xi_t^{(k)}\right)\right] = \mathbb{E}\left[\exp\left(\xi_1^{(k)}\right)\right]^t \leq \mathbb{E}\left[\exp\left(\hat{\xi}_1^{(k)}\right)\right]^t = \mathbb{E}\left[\exp\left(\hat{\xi}_t^{(k)}\right)\right].$$

Hence, for $\pi_t \in [0,\infty)$ as defined in (2.3.2),

$$\mathbb{E}\left[\exp\left(\xi_{\pi_t}^{(k)}\right)\right] = \mathbb{E}\left[\mathbb{E}\left[\exp\left(\xi_1^{(k)}\right)\right]^{\pi_t}\right] \leq \mathbb{E}\left[\mathbb{E}\left[\exp\left(\hat{\xi}_1^{(k)}\right)\right]^{\pi_t}\right] = \mathbb{E}\left[\exp\left(\hat{\xi}_{\pi_t}^{(k)}\right)\right]. \quad (3.1.1)$$

Let $\mathcal{G}$ be the σ-algebra generated by $(T_n)_{n\in\mathbb{N}}$. Then, by the tower property and independence of the Lévy processes from the other components of the decomposition (2.3.2),

$$\mathbb{E}[Y_t] = \mathbb{E}\left[\mathbb{E}\left[\exp\left(\xi_{\pi_t}^{(N_t)} + \sum_{k=0}^{N_t-1}\xi_{\zeta_{k-}}^{(k)}\right)\middle|\,\mathcal{G}\right]\mathbb{E}\left[\exp\left(\sum_{k=0}^{N_t-1}U_k\right)\middle|\,\mathcal{G}\right]\right]. \quad (3.1.2)$$

Consider the first of the two conditional expectations of (3.1.2). Independence and (3.1.1) gives,

$$\begin{aligned}\mathbb{E}\left[\exp\left(\xi_{\pi_t}^{(N_t)} + \sum_{k=0}^{N_t-1}\xi_{\zeta_{k-}}^{(k)}\right)\middle|\,\mathcal{G}\right] &= \mathbb{E}\left[\exp\left(\xi_{\pi_t}^{(N_t)}\right)\middle|\,\mathcal{G}\right]\prod_{k=0}^{N_t-1}\mathbb{E}\left[\exp\left(\xi_{\zeta_{k-}}^{(k)}\right)\middle|\,\mathcal{G}\right]\\ &\leq \mathbb{E}\left[\exp\left(\hat{\xi}_{\pi_t}^{(N_t)}\right)\middle|\,\mathcal{G}\right]\prod_{k=0}^{N_t-1}\mathbb{E}\left[\exp\left(\hat{\xi}_{\zeta_{k-}}^{(k)}\right)\middle|\,\mathcal{G}\right]\\ &= \mathbb{E}\left[\exp\left(\hat{\xi}_{\pi_t}^{(N_t)} + \sum_{k=0}^{N_t-1}\hat{\xi}_{\zeta_{k-}}^{(k)}\right)\middle|\,\mathcal{G}\right].\end{aligned}$$

Moreover, by definition, $t = \pi_t + \sum_{k=0}^{N_t-1}\zeta_k$ and since the increments of a Lévy process are i.i.d.,

$$\mathbb{E}\left[\exp\left(\xi_{\pi_t}^{(N_t)} + \sum_{k=0}^{N_t-1}\xi_{\zeta_{k-}}^{(k)}\right)\middle|\,\mathcal{G}\right] \leq \mathbb{E}\left[\exp\left(\hat{\xi}_t\right)\right].$$

From its definition, $\hat{\xi}$ has finite exponential moments if $\xi^{(\alpha)}$ does for all $\alpha \in E$, and so, the same holds for ξ.

Now, consider the second conditional expectation of (3.1.2) and notice that $\{N_t : t \geq 0\}$ is $\mathcal{G}$ measurable. Hence, it follows that

$$\mathbb{E}\left[\exp\left(\sum_{k=0}^{N_t-1}U_k\right)\middle|\,\mathcal{G}\right] = \prod_{k=0}^{N_t-1}\mathbb{E}\left[\exp\left(U_k\right)\,|\,\mathcal{G}\right] \leq \prod_{k=0}^{N_t}\hat{V},$$

where, $\hat{V} := \max_{\alpha,\beta\in E} \mathbb{E}\left[\exp(U_{\alpha,\beta})\right] \geq \max_{k\in\mathbb{N}} \mathbb{E}\left[\exp\left(U_k\right) \mid \mathcal{G}\right]$. Notice that $\hat{V} < \infty$, if and only if, $\mathbb{E}\left[\exp\left(U_{\alpha,\beta}\right)\right] < \infty$, for all $\alpha, \beta \in E$. Recall from Lemma 3.1.1, that $N_t \leq \eta_t$, for all $t > 0$, where $\{\eta_t : t \geq 0\}$ is a Poisson process of rate $\lambda := \max_{\alpha\in E} q_\alpha$. Therefore, by standard results for Poisson processes,

$$\mathbb{E}\left[\exp\left(\sum_{k=0}^{N_t-1} U_k\right)\right] \leq \mathbb{E}\left[\prod_{k=0}^{\eta_t-1} \hat{V}\right] = \mathbb{E}\left[\hat{V}^{\eta_t}\right] \leq \exp\left(\frac{t(\hat{V}-1)}{\lambda}\right).$$

Hence, $\mathbb{E}[Y_t] \leq \hat{B}^t$, where $\hat{B} := \mathbb{E}\left[\exp\left(\hat{\xi}_1\right)\right] \exp\left((\hat{V}-1)\lambda^{-1}\right)$ is a constant. Clearly, if $\mathbb{E}[\exp(\xi^{(\alpha)})] < \infty$ and $\mathbb{E}[\exp(U_{\alpha,\beta})] < \infty$ for all $\alpha, \beta \in E$, then $\hat{B}$ is finite. Thus, this is a sufficient condition for $\mathbb{E}\left[Y_t\right] < \infty$, for all $t \geq 0$.

To prove necessity, suppose one of the components, $\xi^{(\alpha)}$ or $U_{\alpha,\beta}$ for some $\alpha, \beta \in E$, of the decomposition (2.3.2) fails to have exponential moments. Then, with positive probability, that component appears in the product (3.1.2) and hence $\mathbb{E}[Y_t] = \infty$ for all $t > 0$.

$\square$

Now consider the conditions required for *uniform integrability* of Y. The following adaptation of [54, pp 174, Section 2, Lemma 1] from Lévy processes to MAPs will be needed in the proof of the uniform integrability statement of Theorem 3.1.1.

Lemma 3.1.3

For any $T > 0$ and $0 < u_0 < u$, the following bound holds:

$$\mathbb{P}\left(\sup_{t\in[0,T]} \xi_t \geq u\right) \leq \frac{\mathbb{P}\left(\xi_T > u - u_0\right)}{\min_{\alpha\in E} \mathbb{P}_\alpha\left(\inf_{s\in[0,T]} \xi_s \geq -u_0\right)}.$$

Proof

Consider the stopping time $S_u := \inf\{t \geq 0 | \xi_t > u\}$. Since ξ is càdlàg , $\xi_{S_u} \geq u$, hence

$$\begin{aligned}
\mathbb{P}(S_u < T; \xi_T < u - u_0) &\leq \mathbb{P}(S_u < T; \xi_T - \xi_{S_u} < -u_0) \\
&\leq \mathbb{P}\left(S_u < T; \inf_{s\in[S_u,S_u+T]} (\xi_s - \xi_{S_u}) < -u_0\right) \\
&= \sum_{\alpha\in E} \mathbb{P}(S_u < T; J_{S_u} = \alpha)\mathbb{P}_\alpha\left(\inf_{s\in[0,T]} \xi_s < -u_0\right).
\end{aligned}$$

This leads to the inequality

$$\begin{aligned}
\mathbb{P}(S_u < T) &\leq \mathbb{P}(\xi_T \geq u - u_0) + \mathbb{P}(S_u < T; \xi_T < u - u_0) \\
&\leq \mathbb{P}(\xi_T \geq u - u_0) + \sum_{\alpha\in E} \mathbb{P}(S_u < T; J_{S_u} = \alpha)\mathbb{P}_\alpha\left(\inf_{s\in[0,T]} \xi_s < -u_0\right),
\end{aligned}$$

which can be rearranged to give,

$$\sum_{\alpha\in E}\mathbb{P}(S_u<T;J_{S_u}=\alpha)\mathbb{P}_\alpha\left(\inf_{s\in[0,T]}\xi_s\geq -u_0\right)\leq \mathbb{P}(\xi_T\geq u-u_0).$$

Then, the result of the lemma follows from the inequality

$$\mathbb{P}\left(\sup_{t\in[0,T]}\xi_t\geq u\right)=\sum_{\alpha\in E}\mathbb{P}\left(S_u<T;J_{S_u}=\alpha\right)\leq \frac{\mathbb{P}(\xi_T\geq u-u_0)}{\min_{\alpha\in E}\mathbb{P}_\alpha\left(\inf_{s\in[0,T]}\xi_s\geq -u_0\right)}.$$

□

It is known that there are no *strictly locally integrable* Lévy processes [45, Exercise 29, pp 49]. By a straightforward adaptation of the proof, the same is true for exponential of Lévy processes. A corresponding result for Y is derived in the following lemma. Similarly, it is also shown that Y can not be a *strictly local martingale*.

Lemma 3.1.4
If $\{Y_t : t\geq 0\}$ is locally integrable, then it is also integrable. Moreover, if Y is a local martingale, then it is also a true martingale.

Proof
Suppose Y is locally integrable and let $\{\tau_n\}_{n\in\mathbb{N}}$ be a localising sequence of stopping times. Define a new stopping time $\tau := \min_{n\in\mathbb{N}}\{\tau_n : \tau_n > T_1\}$.

First, suppose $J_0=\alpha\in E$ and let $Y_t^{(\tau)} := Y_{\tau\wedge t}$, for $t\geq 0$, be the process Y stopped at τ. Since T_1 is also a stopping time, by local integrability $\mathbb{E}[Y_{\tau\wedge T_1}]<\infty$. However, since $T_1<\tau$ and J is a Markov chain, for each $\alpha\in E$ we have

$$\mathbb{E}_\alpha\left[Y_{\tau\wedge T_1}\right]=\mathbb{E}_\alpha\left[Y_{T_1}\right]=\mathbb{E}\left[\exp\left(\xi_{T_1}^{(\alpha)}\right)\right]\sum_{\beta\in E\setminus\{\alpha\}}\mathbb{E}\left[\exp\left(U_{\alpha,\beta}\right)\right]\frac{q_{\alpha,\beta}}{q_\alpha}.$$

Thus, if $\mathbb{E}\left[\exp\left(\xi_{T_1}^{(\alpha)}\right)\right]=\infty$ or $\mathbb{E}\left[\exp\left(U_{\alpha,\beta}\right)\right]=\infty$ for any $\beta\in E$, then $\mathbb{E}_\alpha[Y_{\tau\wedge T_1}]=\infty$, contradicting local integrability.

Now consider $J_0=\beta\neq\alpha$. Then, let $S=\min_{n\in\mathbb{N}}\{J_{T_n}=\alpha\}$ and notice that for all $T>0$,

$$\mathbb{E}_\beta\left[Y_T;T>S\right]=\int_0^T\mathbb{E}_\beta\left[Y_s|S=s\right]\mathbb{E}_\alpha[Y_{T-s}]\mathbb{P}_\beta(S\in ds).$$

Then, since J is irreducible, $\mathbb{P}_\beta(S<T)>0$. Thus, Y is locally integrable with respect to $\mathbb{P}_\beta$ only if it is locally integrable with respect to $\mathbb{P}_\alpha$ also. Hence, for any initial distribution of J, the process Y is locally integrable, only if $\mathbb{E}\left[\exp\left(\xi^{(\alpha)}\right)\right]<\infty$ and $\mathbb{E}\left[\exp\left(U_{\alpha,\beta}\right)\right]<\infty$, for all $\alpha, \beta \in E$. However, by Lemma 3.1.2, these are precisely the conditions for Y to be integrable. Hence, Y is not strictly locally integrable.

Now consider the second claim and suppose that Y is a local martingale. For ease of notation, for each $T > 0$ let $\bar{\xi}_T := \sup_{t\in[0,T]} \xi_t$ and $\bar{Y}_T := \sup_{t\in[0,T]} Y_t$.

Then, for $K > 1$, by integration by parts,

$$\begin{aligned}
&\mathbb{E}\left[\exp\left(\bar{\xi}_T\right); \bar{\xi}_T > \log(K)\right] \\
&\quad = \lim_{x\to\infty} \left\{-\exp(x)\mathbb{P}\left(\bar{\xi}_T \geq x\right)\right\} + K\mathbb{P}\left(\bar{\xi}_T \geq \log(K)\right) + \int_{\log(K)}^{\infty} \exp(x)\mathbb{P}(\bar{\xi}_T \geq x)dx \\
&\quad = \lim_{x\to\infty} \left\{-x\mathbb{P}\left(\bar{\xi}_T \geq \log(x)\right)\right\} + K\mathbb{P}\left(\bar{\xi}_T \geq \log(K)\right) + \int_{K}^{\infty} \mathbb{P}\left(\bar{\xi}_T \geq \log(x)\right)dx.
\end{aligned}$$

Since $K > 1$, we can choose $u_0 \in (0, \log(K))$. Then, by applying Lemma 3.1.3,

$$\begin{aligned}
H(u_0)\mathbb{E}\left[\bar{Y}_t; \bar{Y}_t > K\right] \leq &\lim_{x\to\infty} -x\mathbb{P}\left(\bar{\xi}_T \geq \log(x) - u_0\right) \\
&+ K\mathbb{P}(\xi_T \geq \log(K) - u_0) + \int_{K}^{\infty} \mathbb{P}(\xi_T \geq \log(x) - u_0)dx,
\end{aligned}$$

where $H(u_0) := \min_{\alpha\in E} \mathbb{P}_\alpha\left(\inf_{s\in[0,T]} \xi_s \geq -u_0\right)$ and taking K and u_0 sufficiently large ensures $H(u_0) > 0$. Moreover, since Y_T is integrable, $\lim_{x\to\infty} -x\mathbb{P}\left(\bar{\xi}_T \geq \log(x) - u_0\right) = 0$. Then, rewriting in terms of Y_T gives

$$H(u_0)\mathbb{E}\left[\bar{Y}_T; \bar{Y}_T > K\right] \leq K\mathbb{P}(Y_T \geq Ke^{-u_0}) + e^{u_0}\int_{Ke^{-u_0}}^{\infty} \mathbb{P}(Y_T \geq w)dw \tag{3.1.3}$$

$$= e^{u_0}\mathbb{E}\left[Y_T; Y_T \geq Ke^{-u_0}\right]. \tag{3.1.4}$$

Hence, $\mathbb{E}\left[\bar{Y}_T\right] \leq K + \frac{e^{u_0}}{H(u_0)}\mathbb{E}\left[Y_T\right] < \infty$, where finiteness is due to the integrability of Y, that follows from the fact Y can not be strictly locally integrable.

Then, since $\tau_n \to \infty$ as $n \to \infty$ a.s., it follows that $Y_T^{(\tau_n)} \to Y_T$ as $n \to \infty$ a.s.. For each, $n \in \mathbb{N}$, the inequality $Y_T^{(\tau_n)} \leq \bar{Y}_T$ holds and by the above argument $\bar{Y}_T$ is integrable. Hence, by the dominated convergence theorem, for each $t \in [0, T]$, a.s.

$$\mathbb{E}\left[Y_T | \mathcal{F}_t\right] = \lim_{n\to\infty} \mathbb{E}\left[Y_T^{(\tau_n)} \middle| \mathcal{F}_t\right] = \lim_{n\to\infty} Y_t^{(\tau_n)} = Y_t,$$

and so Y is a true martingale.

□

We can now proceed with the proof of Theorem 3.1.1.

Proof of Theorem 3.1.1

Following Remark 3.1.1, we will show the result for the case $p = 1$. The equivalence of (1) and (4) and then of (1) and (2) is given by Lemma 3.1.2. The equivalence of (1) and (3) is given by Lemma 3.1.4. The equivalence of (4) and (5) follows from the result for Lévy processes given in [48, pp 159, Chapter 5, Section 25, Theorem 25.3]. The equivalence of (4) and (6) follows from the definition of F in (2.3.4).

Now consider the final statement regarding *uniformly integrability*. It is known that there is a real left eigenvector, $h \in \mathbb{R}^{|E|}$, of $F(1)$, corresponding to the principal eigenvalue, $\lambda(1)$, which has non-negative entries and is such that $\sum_\alpha h_\alpha = 1$ (for example see [51, pp 5, Section 1, Proposition 1.3]). Thus, h may be used as the initial distribution over E of J. Moreover, h is also a left eigenvector of $e^{tF(1)}$, corresponding to the eigenvalue $e^{t\lambda(1)}$. Let $\mathbb{P}_h$ and $\mathbb{E}_h$ denote the probability measure and corresponding expectation, respectively, when J has an initial distribution given by h.

For the case $\kappa \leq 1$, we first show that Y is not uniformly integrable with respect to $\mathbb{P}_h$ and then use this to prove that Y is not uniformly integrable with respect to any initial distribution of J. In this case, $\lambda(1) \geq 0$, hence h is a left eigenvector of $e^{tF(1)}$, corresponding to the eigenvalue $e^{t\lambda(1)} \geq 1$. Thus,

$$\mathbb{E}_h\left[Y_t\right] = \sum_{\alpha\in E}\sum_{\beta\in E} h_\beta \mathbb{E}_\beta\left[Y_t; J_t = \alpha\right] = \sum_{\alpha\in E}\left(he^{tF(1)}\right)_\alpha = e^{t\lambda(1)}\sum_{\alpha\in E} h_\alpha \geq 1,$$

for all $t \geq 0$. However, under Cramér's condition it is known that $\lim_{t\to\infty} t^{-1}\xi_t = \lambda'(0) < 0$ almost surely and hence also in probability (for example see [4, pp 313, Chapter XI, Section 2, Corollary 2.8] and [35, pp 9, Section 2.3]). By choosing $\epsilon \in (0, -\lambda'(0))$, there exists $\tau_1 > 0$ such that $\exp(t(\lambda'(0) + \epsilon)) < \frac{1}{2}$, for all $t > \tau_1$. Moreover, by convergence in probability, for all $\delta > 0$ there exists $\tau_2 > 0$ such that, for all $t > \tau_2$,

$$\delta \geq \mathbb{P}_h\left(t^{-1}\xi_t - \lambda'(0) > \epsilon\right) = \mathbb{P}_h(Y_t > \exp(t(\lambda'(0) + \epsilon))),$$

and so for $t > \max(\tau_1, \tau_2)$,

$$\mathbb{P}_h\left(Y_t > \frac{1}{2}\right) \leq \mathbb{P}_h\left(Y_t > \exp\left(t\left(\lambda'(0) + \epsilon\right)\right)\right) \leq \delta,$$

that is, $\mathbb{P}_h(Y_t > 1/2) \to 0$ as $t \to \infty$.

Now suppose for contradiction that Y is uniformly integrable with respect to $\mathbb{P}_h$. Then, for all $\gamma > 0$, there exists $K > 0$ such that $\mathbb{E}_h\left[Y_t; Y_t > K\right] < \gamma$, for $t > 0$. Hence, for all $t \geq 0$,

$$\begin{aligned}1 \leq \mathbb{E}_h[Y_t] &= \mathbb{E}_h\left[Y_t; Y_t < \frac{1}{2}\right] + \mathbb{E}_h\left[Y_t; \frac{1}{2} \leq Y_t \leq K\right] + \mathbb{E}_h\left[Y_t; Y_t > K\right]\\ &\leq \frac{1}{2} + K\mathbb{P}_h\left(Y_t \geq \frac{1}{2}\right) + \gamma.\end{aligned}$$

By taking the limit as $t \to \infty$ and using the above result, we obtain $1 \leq \frac{1}{2} + \gamma$, which is clearly a contradiction for $\gamma < \frac{1}{2}$. Thus, in the case $\kappa \leq 1$, Y isn't uniformly integrable with respect to $\mathbb{P}_h$.

However, if Y is not uniformly integrable with respect to $\mathbb{P}_h$, then there must exist an $\alpha \in E$, such that Y is not uniformly integrable with respect to $\mathbb{P}_\alpha$. Now consider any $\beta \in E$. Then, for any $K > 0$ and $t > 1$,

$$\begin{aligned}\mathbb{E}_\beta\left[Y_t; Y_t > K\right] &\geq \mathbb{E}_\beta\left[\exp\left(\xi_t\right); \xi_t > \log(K); J_1 = \alpha\right] \\ &\geq \mathbb{E}_\beta\left[\exp(\xi_1)\hat{\mathbb{E}}_\alpha\left[\exp\left(\hat{\xi}_{t-1}\right); \hat{\xi}_{t-1} > \log(K) - \xi_1\right]; J_1 = \alpha\right],\end{aligned}$$

where $\left(\hat{J}, \hat{\xi}\right)$ is an independent and identically distributed copy of (J, ξ), with corresponding expectation $\hat{\mathbb{E}}$. However, since Y is not uniformly integrable with respect to $\mathbb{P}_\alpha$, there exists $\delta > 0$ such that, $\limsup_{t\to\infty} \mathbb{E}_\alpha\left[\exp(\xi_t); \xi_t > \log(K)\right] > \delta$, for all $K > 0$. Thus, $\limsup_{t\to\infty} \mathbb{E}_\beta\left[Y_t; Y_t > K\right] \geq \delta\mathbb{E}_\beta\left[\exp\left(\xi_1\right); J_1 = \alpha\right]$. Then, since $\mathbb{E}_\beta\left[\exp(\xi_1); J_1 = \alpha\right] > 0$, we don't have uniform integrability of Y with respect to $\mathbb{P}_\beta$ for any $\beta \in E$. Hence, Y is not uniformly integrable for any initial distribution of J, whenever $\kappa \leq 1$.

Now suppose that $\kappa > 1$. Then, $\lambda(1) < 0$ and it follows that

$$\mathbb{E}_h[Y_t; Y_t > K] \leq \mathbb{E}_h[Y_t] = \sum_{\alpha\in E}\sum_{\beta\in E} h_\beta \mathbb{E}_\beta\left[Y_t; J_t = \alpha\right] = \sum_{\alpha\in E}\left(he^{tF(1)}\right)_\alpha = \sum_{\alpha\in E} h_\alpha e^{t\lambda(1)} \to 0,$$

as $t \to \infty$. Moreover, since each entry of h is strictly positive, $\mathbb{E}_\alpha[Y_t; Y_t > K] \to 0$ as $t \to \infty$, for all $\alpha \in E$. Hence, for all $\epsilon > 0$, there exists $T > 0$ such that $\mathbb{E}_\alpha[Y_t; Y_t > K] < \epsilon$, for all $t > T$, $K > 0$ and $\alpha \in E$.

We now consider $t \in [0, T]$ and $K > 1$. Taking the limit as $K \to \infty$ in (3.1.3) and letting $\bar{Y}_T := \sup_{t\in[0,T]} Y_t$, gives

$$\lim_{K\to\infty}\sup_{t\in(0,T)} \mathbb{E}\left[Y_t; Y_t > K\right] \leq \lim_{K\to\infty} \mathbb{E}\left[\bar{Y}_T; \bar{Y}_T > K\right] \leq \lim_{K\to\infty} \frac{e^{u_0}\mathbb{E}\left[Y_T; Y_T \geq Ke^{-u_0}\right]}{H(u_0)} = 0,$$

since Y_T is integrable, for some u_0 sufficiently large. Combined with the result for $t > T$, this implies $\{Y_t : t \geq 0\}$ is uniformly integrable. $\square$

3.2 Finiteness of the Exponential Functional

Throughout this section we consider the Lamperti-Kiu setting, where J has state space $E = \{+, -\}$ and $Y_t = J_t \exp(\xi_t)$, $t \geq 0$ is a Lamperti-Kiu process. Then we consider the standard and signed exponential functionals, A_∞ and B_∞, respectively.

If at least one of $\mathbb{E}\left[\max(\xi_{T_2},0)\right] < \infty$ and $\mathbb{E}\left[\max(-\xi_{T_2},0)\right] < \infty$, define $K \in \mathbb{R}\cup\{-\infty,+\infty\}$ by

$$K := \frac{\mathbb{E}[\xi_{T_2}]}{\mathbb{E}[T_2]} = \frac{q_-}{q_+ + q_-}\left(\mathbb{E}\left[\xi_1^{(+)}\right] + q_+\mathbb{E}[U_+]\right) + \frac{q_+}{q_+ + q_-}\left(\mathbb{E}\left[\xi_1^{(-)}\right] + q_-\mathbb{E}[U_-]\right), \quad (3.2.1)$$

where K may take the values $+\infty$ and $-\infty$. If both $\mathbb{E}[\max(\xi_{T_2},0)] = \infty$ and $\mathbb{E}[\max(-\xi_{T_2},0)] = \infty$, then K is undefined.

A Lamperti-Kiu process Y will be called *degenerate* if Y is such that $\limsup_{t\to\infty} |\xi_t| < \infty$. This can be shown to be equivalent to the case that either Y has a finite lifetime or $\xi_t^{(\pm)} \equiv 0$ for all $t \geq 0$ and $U_+ = -U_-$ is deterministic. In this case,

$$Y_t^{(x)} = \begin{cases} x, & \text{if } T_{2k} \quad \leq t < T_{2k+1} \text{ for some } k \in \mathbb{N}_0; \\ x\exp(U^{\mathrm{sgn}(x)}), & \text{if } T_{2k+1} \leq t < T_{2k+2} \text{ for some } k \in \mathbb{N}_0; \end{cases}$$

for all $t \geq 0$ and $x \in \mathbb{R}^*$.

When K is defined and Y has an infinite lifetime, from [4, pp 214, Proposition 2.10] and [4, pp 313, Corollary 2.8], it is known that $\lim_{t\to\infty} t^{-1}\xi_t = K$ a.s.. Moreover, it is shown that if $K = 0$ and Y is non-degenerate, then $\lim_{t\to\infty} t^{-1}\xi_t = 0$, $\liminf_{t\to\infty} \xi_t = -\infty$ and $\limsup_{t\to\infty} \xi_t = \infty$.

To consider the case where K is undefined, a result is needed for MAPs which is analogous to [22, pp 372, Theorem 2]. First, define the functions

$$m_-(x) := \int_{-x}^{0} \mathbb{P}(\xi_{T_2} \leq y)dy, \qquad m_+(x) := \int_0^x \mathbb{P}(\xi_{T_2} > y)dy,$$

for $x \in \mathbb{R}^+$ and

$$I_+ := \int_0^\infty \frac{x}{m_-(x)}\mathbb{P}(\xi_{T_2} \in dx), \qquad I_- := \int_{-\infty}^0 \frac{|x|}{m_+(|x|)}\mathbb{P}(\xi_{T_2} \in dx).$$

Then, the long term behaviour of $\{\xi_t : t \geq 0\}$ is described by the following lemma.

Lemma 3.2.1

Suppose K is undefined. Then, at least one of I_+ and I_- equals infinity and almost surely:

(i) $\limsup_{t\to\infty} t^{-1}\xi_t = \infty$ *if and only if* $I_+ = \infty$;

(ii) $\lim_{t\to\infty} t^{-1}\xi_t = \infty$ *if and only if* $I_+ = \infty$ *and* $I_- < \infty$;

(iii) $\liminf_{t\to\infty} t^{-1}\xi_t = -\infty$ *if and only if* $I_- = \infty$;

(iv) $\lim_{t\to\infty} t^{-1}\xi_t = -\infty$ *if and only if* $I_+ < \infty$ *and* $I_- = \infty$.

Proof

Consider the sequence $\{\xi_{T_{2n}}\}_{n\in\mathbb{N}}$ as the random walk

$$\xi_{T_{2n}} = \sum_{k=1}^{n} \left(\xi_{T_{2k}} - \xi_{T_{2k-2}}\right), \qquad n \in \mathbb{N},$$

and notice that $\xi_{T_{2n}} - \xi_{T_{2(n-1)}} \stackrel{\mathcal{L}}{=} \xi_{T_2}$ has an undefined mean, for each $n \in \mathbb{N}$. Then, by applying Erickson's theorem for random walks [22, pp 372, Theorem 2] and the remark that follows it, either $I_+ = \infty$ or $I_- = \infty$ or both hold, proving the first statement of the lemma. Furthermore, the following statements hold:

(1) $\limsup_{n\to\infty} n^{-1}\xi_{T_{2n}} = \infty$ a.s. if and only if $I_+ = \infty$;

(2) $\lim_{n\to\infty} n^{-1}\xi_{T_{2n}} = \infty$ a.s. if and only if $I_+ = \infty$ and $I_- < \infty$;

(3) $\liminf_{n\to\infty} n^{-1}\xi_{T_{2n}} = -\infty$ a.s. if and only if $I_- = \infty$;

(4) $\lim_{n\to\infty} n^{-1}\xi_{T_{2n}} = -\infty$ a.s. if and only if $I_+ < \infty$; and $I_- = \infty$;

and similar statements hold for $\{T_{2n+1}\}_{n\in\mathbb{N}}$.

Since $\mathbb{E}[T_2] < \infty$, it is immediate that if $\limsup_{n\to\infty} n^{-1}\xi_{T_{2n}} = \infty$ then $\limsup_{t\to\infty} t^{-1}\xi_t = \infty$ also and if $\liminf_{n\to\infty} n^{-1}\xi_{T_{2n}} = -\infty$ then $\liminf_{t\to\infty} t^{-1}\xi_t = -\infty$. Hence, the "if" direction of statements (i) and (iii) holds. To prove the "only if" direction of (i) and (iii) we must first prove (ii) and (iv).

Consider (iv) and notice that the "only if" direction is an immediate consequence of statement (4) above. Now, suppose $I_+ < \infty$ and $I_- = \infty$. Then, $\limsup_{n\to\infty} n^{-1}\xi_{T_{2n}} = -\infty$ and $\limsup_{n\to\infty} n^{-1}\xi_{T_{2n+1}} = -\infty$, hence $\limsup_{n\to\infty} n^{-1}\xi_{T_n} = -\infty$ also. Suppose for a contradiction there exists an $R > 0$ such that $\limsup_{t\to\infty} t^{-1}\xi_t > -R > -\infty$. Since $\lim_{n\to\infty} n^{-1}T_n = \frac{1}{2}\mathbb{E}[T_2]$ and $\limsup_{n\to\infty} n^{-1}\xi_{T_n} = -\infty$, there a.s. exists some $N \in \mathbb{N}$ such that $n > N$ implies $T_n > 1$ and $\frac{\xi_{T_n}}{T_n} < -2R$.

Define sequences $\{\tau_n\}_{n\in\mathbb{N}}$ of times and $\{x_n\}_{n\in\mathbb{N}}$ of values such that, for each $n \in \mathbb{N}$,

$$\tau_n = \sup\left\{t \in [T_n, T_{n+1}) \,:\, \xi_t = \sup_{s\in[T_n,T_{n+1})} \xi_s\right\} \qquad \text{and} \qquad x_n = \sup\left\{\xi_t \,:\, t \in [T_n, T_{n+1})\right\}.$$

Since $\limsup_{t\to\infty} t^{-1}\xi_t > -R$, there is an increasing sequence of times $\{s_n\}_{n\in\mathbb{N}}$, such that $s_n^{-1}\xi_{s_n} > -R$ for each $n \in \mathbb{N}$ and $\lim_{n\to\infty} s_n = \infty$. Then, take a subsequence $\{s'_n\}_{n\in\mathbb{N}}$, such that there is at most one element of the sequence $\{s'_n\}_{n\in\mathbb{N}}$ in each interval $[T_m, T_{m+1}]$ and $J_{s'_n}$ is constant.

Let $\{\tau_{k_n}\}_{n\in\mathbb{N}}$ be a subsequence of $\{\tau_n\}_{n\in\mathbb{N}}$, such that $k_n > N$ and $s'_n \in [T_{k_n}, T_{k_n+1}]$ for each $n \in \mathbb{N}$. Then, for each $n \in \mathbb{N}$, we have

$$x_{k_n} \geq \xi_{s'_n} > -Rs'_n \geq -RT_{k_n+1},$$

whilst $\xi_{T_{k_n+1}} < -2RT_{k_n+1}$ and $T_{k_n+1} > 1$, therefore

$$x_{k_n} - \xi_{T_{k_n}+1} > RT_{k_n+1} > R.$$

However, for each $n \in \mathbb{N}$, we have that $\{\xi_{T_{k_n}+t} - \xi_{T_{k_n}} : t < T_{k_n+1} - T_{k_n}\}$ is a Lévy process. Hence, by splitting at the last time of the maximum [5, pp 160, Chapter VI, Theorem 5], $x_{k_n} - \xi_{T_{k_n+1}}$ is independent of x_{k_n} and has the same distribution as $x_0 - \xi_{T_1}$, hence its distribution doesn't depend on R. This contradicts the fact that it only has support on (R, ∞). Hence, $I_+ < \infty$ and K undefined imply that $\lim_{t\to\infty} t^{-1}\xi_t = -\infty$, thus, the "if" direction of (iv) holds. By applying similar arguments to $-\xi_t$, statement (iii) of the lemma also holds.

To prove the "only if" direction of (i), suppose $I_+ < \infty$. Then, by [22, pp 372, Theorem 2], since K is undefined, $I_- = \infty$. Hence, $\lim_{t\to\infty} t^{-1}\xi_t = -\infty$, by statement (iv), and so, $\limsup_{t\to\infty} t^{-1}\xi_t = \infty$ only if $I_+ = \infty$. The argument for (iii) is analogous. □

The following theorem shows that the convergence and finiteness of A_∞ and B_∞ is fully characterised by K, when this is defined, and by the finiteness of I_+, otherwise.

Theorem 3.2.1 (Finiteness of the Exponential Functional)
Consider the Lamperti-Kiu case. Then, A_∞ converges if and only if B_∞ converges. Moreover, A_∞ and B_∞ converge if and only if either K is defined and $K < 0$ or K is undefined and $I_+ < \infty$.

Before proving Theorem 3.2.1, we prove the following preliminary lemma.

Lemma 3.2.2
If $\limsup_{n\to\infty} \xi_{T_{2n}} = \infty$ a.s., then both A_∞ and B_∞ diverge a.s..

Proof
If $\limsup_{n\to\infty} \xi_{T_{2n}} = \infty$ a.s., then there exists a strictly increasing sequence $\{\tau_n\}_{n\in\mathbb{N}} \subset \mathbb{N}$, such that $\exp\left(\xi_{T_{2\tau_n}}\right) \geq 1$ for all $n \in \mathbb{N}$ a.s.. First, by considering A_∞ and using the Markov property for the second inequality,

$$A_\infty \geq \sum_{n=0}^{\infty} \exp\left(\xi_{T_{2\tau_n}}\right) \int_{T_{2\tau_n}}^{T_{2\tau_n+2}} \exp\left(\xi_t - \xi_{T_{2\tau_n}}\right) dt \geq \sum_{n=0}^{\infty} \int_0^{\hat{T}_2^{(n)}} \exp\left(\hat{\xi}_t^{(n)}\right) dt,$$

where $\left\{\hat{\xi}^{(n)}\right\}_{n\in\mathbb{N}}$ and $\left\{\hat{T}_2^{(n)}\right\}_{n\in\mathbb{N}}$ are sequences of i.i.d. copies of ξ and T_2, respectively. However, the last term of the above expression is a sum of strictly positive i.i.d. terms and so the series must diverge.

Similarly, B_∞ converges only if the sum $\sum_{n=0}^{\infty}\int_{T_{2n}}^{T_{2n+2}} J_t \exp(\xi_t)dt$ converges, which implies convergence to zero of the subsequence

$$\begin{aligned}\left|\int_{T_{2\tau_n}}^{T_{2\tau_n+2}} J_t \exp(\xi_t)dt\right| &= \exp\left(\xi_{T_{2\tau_n}}\right)\left|\int_{T_{2\tau_n}}^{T_{2\tau_n+2}} J_t \exp\left(\xi_t - \xi_{T_{2\tau_n}}\right)dt\right| \\ &\geq \left|\int_{T_{2\tau_n}}^{T_{2\tau_n+2}} J_t \exp\left(\xi_t - \xi_{T_{2\tau_n}}\right)dt\right|.\end{aligned}$$

Then, using the Markov property, B_∞ converges to zero only if

$$\left|\int_0^{\hat{T}_2^{(n)}} \hat{J}_t^{(n)} \exp\left(\hat{\xi}_t^{(n)}\right) dt\right| \to 0 \qquad \text{a.s.},$$

as $n \to \infty$. This convergence is impossible, since it is an i.i.d. sequence which doesn't converge to zero in distribution. □

Proof of Theorem 3.2.1

The proofs for the cases that K is defined and undefined are given separately.

(1) K defined:

Suppose that K is defined and first consider $K < 0$. Then, recall $\lim_{t\to\infty} t^{-1}\xi_t = K$ as $t \to \infty$. If $-\infty < K < 0$, let $k = \frac{1}{2}K$, whilst if $K = -\infty$, let $k = -1$. Then, a.s. there exists a finite $T \geq 0$, such that $\xi_t < kt < 0$ for all $t > T$. Thus, $A_\infty \leq \int_0^T \exp(\xi_t)dt + k^{-1}e^{kT} < \infty$ a.s. and, by absolute convergence, it is then immediate that B_∞ also converges a.s..

Next, we consider the case that either $K > 0$ or $K = 0$ and Y is non-degenerate. Then, by [30, pp 167, Chapter 9, Proposition 9.14], $\limsup_{n\to\infty} T_{2n} = \infty$ a.s. and so the result follows from Lemma 3.2.2. If $K = 0$ and Y is degenerate, then

$$\xi_t = \begin{cases} 0, & \text{if } T_{2k} \leq t < T_{2k+1} \text{ for } k \in \mathbb{N}_0; \\ U_{J_0}, & \text{if } T_{2k+1} \leq t < T_{2k+2} \text{ for } k \in \mathbb{N}_0; \end{cases}$$

hence, for all $t \geq 0$,

$$e^{\xi_t} \geq \min\left(1, \exp\left(U_{J_0}\right)\right) =: V > 0,$$

and so $A_\infty = \infty$ a.s.. Also, B_∞ can be written as the sum

$$B_\infty = \sum_{n=0}^{\infty}\left(\int_{T_{2n}}^{T_{2n+1}} J_t e^{\xi_t}dt + \int_{T_{2n+1}}^{T_{2n+2}} J_t e^{\xi_t}dt\right) = J_0 \sum_{n=0}^{\infty}\left(\zeta_{2n} - \exp(U_{J_0})\zeta_{2n+1}\right).$$

Then, since $\zeta_{2n} - \exp(U_{J_0})\zeta_{2n+1}$ doesn't converge to zero in distribution, B_∞ must diverge.

(2) K undefined:

Suppose that K is undefined. From [22, pp 372, Theorem 2], we know that if $I_+ = \infty$, then $\limsup_{n\to\infty} \xi_{T_{2n}} = \infty$ a.s. hence, using Lemma 3.2.2, both A_∞ and B_∞ diverge a.s..

If $I_+ < \infty$, then since K is undefined, as a consequence of [22, pp 372, Theorem 2], $I_- = \infty$. Hence, by Lemma 3.2.1, $\limsup_{t\to\infty} t^{-1}\xi_t = -\infty$ a.s.. Then by the argument of case 1. above both A_∞ and B_∞ converge a.s..

□

Chapter 4

Mellin Transform of the Exponential Functional

As was discussed in Section 2.2, a successful approach to the study of the exponential functional of a Lévy process has been to consider the *Mellin transform* (see Appendix A.3) of its density. This is done by finding solutions to the recurrence relation (2.2.2), within a suitable space of functions. Here, a similar approach is taken to finding the Mellin transform of the density of the exponential functional of a MAP.

In this chapter, we consider a MAP, (J, ξ), and its exponential functional, A_∞, when J has state space $E = \{+, -\}$. The extension of these results to any finite E should not be too difficult, however, the case of an infinite E would present significant difficulties.

Recall from (2.4.1) that the exponential functional A_∞ satisfies the recurrence relation

$$\mathbf{E}[A_\infty^s] = -s(F(s))^{-1}\mathbf{E}[A_\infty^{s-1}], \qquad \Re(s) \in (\rho + 1, \kappa), \tag{4.0.1}$$

where F is the matrix exponent of the MAP, κ is the extended Cramér's number and $\rho = \inf\{z \in \mathbb{R} : \mathbb{E}[e^{z\xi_1}] < \infty\}$. Thus, we first look for solutions to this equation in the form of *matrix valued generalised Gamma function*s.

Under some assumptions, such a solution and some of its properties are given by Theorem 4.1.1. Then, in Theorem 4.1.2 we show that when it exists, this solution is indeed the Mellin transform of the density of A_∞. Hence, for any MAP, the Mellin transform of the density of A_∞ can be found by showing that the conditions of Theorem 4.1.1 are satisfied. This is done for two classes of MAP in Section 4.2.

4.1 General Results

Suppose $H : \mathbb{C} \to \mathbb{C}^{2\times 2}$ and consider the general problem of finding a matrix valued function $\mathcal{H} : \mathbb{C} \to \mathbb{C}^{2\times 2}$, such that $\mathcal{H}(0) = I$ and $\mathcal{H}(s+1) = H(s)\mathcal{H}(s)$, for all $s \in \mathbb{C}$.

The *Webster products* (also known as generalised Gamma functions) from [53], given by

$$\Gamma_g(s) := \lim_{N\to\infty} g^s(N) \prod_{n=1}^{N} \frac{g(n)}{g(n+x)},$$

satisfy the functional equation $\Gamma_g(s+1) = g(s)\Gamma_g(s)$, for some eventually log-concave function $g : \mathbb{R}^+ \to \mathbb{R}^+$. Using this as inspiration, for each $s \in \mathbb{C}$ and $n \in \mathbb{N}$, let

$$M_n(s) = \left(\prod_{k=0}^{n} H(k+s)^{-1}\right) \begin{pmatrix} H_{+,+}(n)^s & 0 \\ 0 & H_{-,-}(n)^s \end{pmatrix} \left(\prod_{k=n}^{0} H(k)\right),$$

where $H_{+,+}$ and $H_{-,-}$ are the diagonal entries of H. The order of the products is specified by the order of the indices, in particular, $\prod_{k=0}^{n} A_k := A_0 \cdots A_n$ and $\prod_{k=n}^{0} A_n \cdots A_0$. We will consider the limit as $n \to \infty$.

To obtain convergence of $\{M_n\}_{n\in\mathbb{N}}$ and certain properties of the limit, we restrict our attention to the strip $\mathcal{S}_{a,b} \subset \mathbb{C}$, for some $0 < a < b$, and suppose that the following assumptions are satisfied.

Assumption 4.1.1

The real numbers $0 < a < b$ and the matrix valued function $H : \mathbb{C} \to \mathbb{C}^{2\times 2}$ are such that:

1. *There exists $u, C_1 > 0$, such that $H_{\alpha,\beta}(s) \le C_1 e^{-u\Re(s)}$, for all $s \in \mathcal{S}_{a,\infty}$ and $\alpha, \beta \in E$ with $\alpha \ne \beta$.*
2. *There exists $C_2 > 0$, such that $|H_{\alpha,\beta}(s)/H_{\gamma,\gamma}(s)| \le C_2 e^{-u\Re(s)}$, for all $s \in \mathcal{S}_{a,\infty}$ and $\alpha, \beta, \gamma \in \{+,-\}$ with $\alpha \ne \beta$.*
3. *There exists $v \in \mathbb{R}$, such that $\lim_{k\to\infty} |\log(H_{+,+}(k)/H_{-,-}(k))| = v$.*
4. *The entries of H are analytic on $\mathcal{S}_{a,\infty}$.*
5. *$H(s)$ is invertible for all $s \in \mathcal{S}_{a,\infty}$.*
6. $\lim_{k\to\infty} \frac{H_{\alpha,\alpha}(k+s)}{H_{\alpha,\alpha}(k)} = 1$ *for all $\alpha \in E$ and $s \in \mathbb{N}$ and uniformly for all $s \in \mathcal{S}_{a,b}$.*

7. *There exists constants $C_3, C_4 > 0$ and $\eta \in \mathbb{N}$ such that $C_3|s|^{-1} \leq |H_{\alpha,\alpha}(s)| \leq C_4|s|^\eta$, for each $\alpha \in \{+,-\}$ and $s \in \mathcal{S}_{a,\infty}$.*

8. *There exist constants $C_5, C_6, t_1, t_2 > 0$ and $N \in \mathbb{N}$, such that for all $s \in \mathcal{S}_{a,b}$, $\alpha \in \{+,-\}$ and $n \in \mathbb{N}$ with $n > N$,*

$$\frac{1}{C_5}\exp(-t_1\pi|\Im(s)|) \leq \left| H_{\alpha,\alpha}(n)^s \prod_{k=1}^{n}\left(1 + \frac{\Delta_s H_{\alpha,\alpha}(k)}{H_{\alpha,\alpha}(k+s)}\right)\right| \leq C_6 \exp(t_2\pi|\Im(s)|),$$

where $2t_1 + t_2 \in (0,2)$. Moreover, the limit as $n \to \infty$ is uniform in $\{s \in \mathcal{S}_{a,b} : |\Im(s)| < T\}$, for all $T \geq 0$.

9. *The constants u and v above satisfy $v - u < 0$.*

10. *For each $k \in \mathbb{N}$, $\sup_{s\in\mathcal{S}_{a,b}} \|H(k+s)^{-1}\| < \infty$.*

Under these assumptions, the existence of a *matrix valued generalised Gamma function* is given by the following theorem.

Theorem 4.1.1 (Existence of Matrix Valued Generalised Gamma Functions)
Under Assumption 4.1.1, the limit $M(s) := \lim_{n\to\infty} M_n(s)$ exists, for all $s \in \mathcal{S}_{a,b}$. Furthermore, M satisfies:

1. *$M(s+1) = H(s)M(s)$, whenever $s, s+1 \in \mathcal{S}_{a,b}$;*
2. *there exists $C_1 > 0$, such that $\|M(s)^{-1}\| \leq C_1 \exp((t_1+t_2)\pi|\Im(s)|)$ for all $s \in \mathcal{S}_{a,b}$;*
3. *there exists $C_2 > 0$, such that $\|M(s)\| \leq C_2 \exp((t_1+t_2)\pi|\Im(s)|)$ for all $s \in \mathcal{S}_{a,b}$;*
4. *each of the entries of M is analytic in $\mathcal{S}_{a,b}$;*

where $t_1, t_2 > 0$ are the constants from Assumption 4.1.1.8.

The proof of this theorem is the focus of Sections 4.1.1 and 4.1.2.

The following result allows for the extension of M outside of $\mathcal{S}_{a,b}$.

Proposition 4.1.1
(i) Suppose that $z \in \mathbb{C} \setminus \mathbb{Z}$ such that $z = s + N$, for some $s \in \mathcal{S}_{a,b}$ and $N \in \mathbb{N}$. Moreover, suppose that $\det(H(s+m)) \neq 0$ for all $m \in \{1, 2, \cdots, N\}$. Then, $M(z)$ is well defined and satisfies

$$M(z) = \left(\prod_{k=1}^{N} H(z-k)\right) M(z-N) = \left(\prod_{k=N-1}^{0} H(s+k)\right) M(s). \qquad (4.1.1)$$

(ii) Similarly, if $z \in \mathbb{C} \setminus \mathbb{Z}$ such that $z = s - N$, for some $s \in \mathcal{S}_{a,b}$ and $N \in \mathbb{N}$, and $\det(F(s-m)) \neq 0$ *for all $m \in \{1, 2, \cdots, N\}$. Then, $M(z)$ is well defined and satisfies*

$$M(z) = \left(\prod_{k=0}^{N-1} H(z+k)^{-1} \right) M(z+N) = \left(\prod_{k=N}^{1} H(s-k)^{-1} \right) M(s). \tag{4.1.2}$$

The proof of this theorem is also given in Section 4.1.1.

Recall that A_∞ is the exponential functional of the MAP (J, ξ) with matrix exponent F. We now consider when a solution to the recurrence relation (4.0.1) gives the Mellin transform of the density of A_∞. Let $D \subset \mathbb{C}$ be the region in which $\mathbb{E}[A_\infty^s] < \infty$ and $D^* := \{s \in D \mid \det(F(s+m)) \neq 0, \forall\, m \in \mathbb{Z}\}$. Notice that $\det(F(0)) = 0$, it follows that $0 \notin D^*$. The following theorem, which extends [34] to MAPs, gives conditions for the solution of (4.0.1) to be the Mellin transform of the density of A_∞.

Theorem 4.1.2 (Verification of Mellin Transforms)
Suppose $a < b - 1$, such that $\mathcal{S}_{a,b} \subset D^*$ and $M : D^* \to \mathbb{C}^{2\times 2}$ is a matrix valued function satisfying, for all $s \in D^*$ and $n \in \mathbb{N}$,

$$M(n) = \prod_{k=n}^{1} \left(\frac{-F(k)}{k} \right)^{-1}, \qquad \text{if } n \in \mathcal{S}_{a,b}, \tag{4.1.3}$$

$$M(-n) = \left(\prod_{k=n-1}^{0} \frac{F(-k)}{k} \right), \qquad \text{if } -n \in \mathcal{S}_{a,b}, \tag{4.1.4}$$

$$M(s+1) = \left(\frac{-F(s+1)}{s+1} \right)^{-1} M(s). \tag{4.1.5}$$

Moreover, suppose there exist real numbers $t_1, t_2 \in \mathbb{R}$ and constants $C_1, C_2 > 0$, such that:

1. $2t_2 + t_1 \in (0, 2)$*;*
2. *for all $s \in \mathcal{S}_{a,b}$, the matrix $M(s)$ is invertible;*
3. *M is analytic on $\mathcal{S}_{a,b}$;*
4. $\|M(s)\| \leq C_1 \exp(t_1 \pi |\Im(s)|)$ *for all $s \in \mathcal{S}_{a,b}$;*
5. $\|M(s)^{-1}\| \leq C_2 \exp(t_2 \pi |\Im(s)|)$ *for all $s \in \mathcal{S}_{a,b}$.*

Then, M has an analytic continuation to D and for all $s \in D$,

$$\mathbf{E}[A_\infty^s] = M(s)\mathbf{m},$$

where

$$\mathbf{m} = \begin{cases} \mathbf{E}[A_\infty^0] = (1,1)^T, & \textit{if } \mathcal{S}_{a,b} \subset \mathbb{C}^+, \\ \mathbf{E}[A_\infty^{-1}] = F'(0)(1,1)^T + F(0)\mathbf{E}[\log A_\infty], & \textit{if } \mathcal{S}_{a,b} \subset \mathbb{C}^-. \end{cases}$$

Section 4.1.3 provides the proof of this theorem.

Remark 4.1.1

Since $0 \notin D^*$ it follows that $0 \notin \mathcal{S}_{a,b}$. Hence, only one of $\mathcal{S}_{a,b} \subset \mathbb{C}^+$ or $\mathcal{S}_{a,b} \subset \mathbb{C}^-$ can be true in the definition of $\mathbf{m}$. That $\mathbf{E}[A_\infty^{-1}] = F'(0)(1,1)^T + F(0)\mathbf{E}[\log A_\infty]$ is shown in [51, Section 1.4.2, Proposition 1.10].

4.1.1 Existence of M

This section provides a proof of the existence of the limits of Theorem 4.1.1. Suppose that Assumption 4.1.1 holds. We first show that the limit $M(s) := \lim_{n\to\infty} M_n(s)$ exists and that, for any $T > 0$, the convergence is uniform in $\{s \in \mathcal{S}_{a,b} : |\Im(s)| < T\}$. For $s \in \mathcal{S}_{a,b}$, $\lambda_+, \lambda_- \in \mathbb{C}$ and $n \in \mathbb{N}$, define the functions

$$L^{(n)}(s; \lambda_+, \lambda_-) := H(n+s)^{-1} \begin{pmatrix} \lambda_+ & 0 \\ 0 & \lambda_- \end{pmatrix} H(n),$$

and

$$L_n(s; \lambda_+, \lambda_-) := \left(\prod_{k=0}^{n} H(k+s)^{-1}\right) \begin{pmatrix} \lambda_+ & 0 \\ 0 & \lambda_- \end{pmatrix} \left(\prod_{k=n}^{0} H(k)\right).$$

We will sometimes use the more compact notation $L^{(n)}(s; \{\lambda_\alpha\}_{\alpha\in E}) := L^{(n)}(s; \lambda_+, \lambda_+)$ and $L_n(s; \{\lambda_\alpha\}_{\alpha\in E}) := L_n(s; \lambda_+, \lambda_+)$. Notice that we can then write M_n in the form $M_n(s) := L_n\left(s; \{H_{\alpha,\alpha}(n)^s\}_{\alpha\in E}\right)$.

For a function $f : \mathbb{C} \to \mathbb{C}$ and $s \in \mathbb{C}$, define the operator $\Delta_s f(k) := f(k) - f(k+s)$.

The following lemma allows us to use an iterative approach to understanding L_n.

Lemma 4.1.1

There exists $C > 0$, such that for all $s \in \mathcal{S}_{a,b}$, $n \in \mathbb{N}$ and $\lambda_+, \lambda_- \in \mathbb{C}$,

$$L_n(s; \{\lambda_\alpha\}_{\alpha\in E}) = L_{n-1}\left(s; \left\{\lambda_\alpha \left(1 + \frac{\Delta_s H_{\alpha,\alpha}(n)}{H_{\alpha,\alpha}(n+s)}\right)\right\}_{\alpha\in E}\right) + C\bar{\lambda}e^{-un}\left(\prod_{k=0}^{n-1} H(k+s)^{-1}\right)\cdot\Omega\cdot\left(\prod_{k=n-1}^{0} H(k)\right),$$

where $\bar{\lambda} \coloneqq \max(\lambda_+, \lambda_-)$ *and* $\Omega \in \mathbb{C}^{2\times 2}$ *is a matrix in which every entry has magnitude less than 1.*

Proof

First, observe that by Assumption 4.1.1.2, for sufficiently large $n \in \mathbb{N}$ and all $s \in \mathcal{S}_{a,b}$,

$$|\det(H(n+s))| = |H_{++}(n+s)H_{--}(n+s)| \left|1 - \frac{H_{+-}(n+s)H_{-+}(n+s)}{H_{++}(n+s)H_{--}(n+s)}\right|.$$

However, $e^{-u(n+s)} < \frac{1}{2}$ for $s \in \mathcal{S}_{a,b} \subset \mathbb{C}^+$ and sufficiently large $n \in \mathbb{N}$. Hence, by Assumption 4.1.1.2

$$\left|\frac{H_{+-}(n+s)H_{-+}(n+s)}{H_{++}(n+s)H_{--}(n+s)}\right| \leq \frac{1}{2},$$

and so, by the triangle inequality and the reverse triangle inequality,

$$\frac{3}{2}\left|H_{++}(n+s)H_{--}(n+s)\right| \geq |\det(H(n+s))| \geq \frac{1}{2}\left|H_{++}(n+s)H_{--}(n+s)\right|. \qquad (4.1.6)$$

We will consider the entries on and off the diagonal separately:

1. Off-diagonal entries

The top-right entry of $L^{(n)}(s; \lambda_+, \lambda_-)$ is given by

$$\frac{\lambda_+ H_{+,-}(n)H_{-,-}(n+s) - \lambda_- H_{+,-}(n+s)H_{-,-}(n)}{\det(H(n+s))}.$$

However, by Assumption 4.1.1.1, there exists a constant $C_1 > 0$, such that

$$\begin{aligned}&|\lambda_+ H_{+,-}(n)H_{-,-}(n+s) - \lambda_- H_{+,-}(n+s)H_{-,-}(n)|\\ &\qquad\qquad \leq C_1\bar{\lambda}e^{-un}|H_{-,-}(n+s)|\left(1 + \left|\frac{H_{-,-}(n)}{H_{-,-}(n+s)}\right|\right),\end{aligned}$$

where $\bar{\lambda} \coloneqq \max_{\alpha\in E}\lambda_\alpha$. Then, by Assumption 4.1.1.6, for sufficiently large $n \in \mathbb{N}$ and for all $s \in \mathcal{S}_{a,b}$,

$$|\lambda_+ H_{+,-}(n)H_{-,-}(n+s) - \lambda_- H_{+,-}(n+s)H_{-,-}(n)| \leq 3C_1\bar{\lambda}e^{-un}|H_{-,-}(n+s)|.$$

Combining this with the bound on the determinant (4.1.6) gives

$$\left|L^{(n)}(s; \lambda_+, \lambda_-)_{+,-}\right| \leq \frac{6C_1\bar{\lambda}e^{-un}|H_{-,-}(n+s)|}{|H_{+,+}(n+s)H_{-,-}(n+s)|} = \frac{6C_1\bar{\lambda}e^{-un}}{|H_{+,+}(n+s)|}.$$

2. Diagonal entries

The top left entry of $L^{(n)}(s; \lambda_+, \lambda_-)$ is given by

$$\frac{\lambda_+ H_{+,+}(n)H_{-,-}(n+s) - \lambda_- H_{+,-}(n+s)H_{-,+}(n)}{H_{+,+}(n+s)H_{-,-}(n+s) - H_{+,-}(n+s)H_{-,+}(n+s)}.$$

Considering its difference from λ_+, we have

$$\frac{\lambda_+ H_{+,+}(n)H_{-,-}(n+s) - \lambda_- H_{+,-}(n+s)H_{-,+}(n)}{H_{+,+}(n+s)H_{-,-}(n+s) - H_{+,-}(n+s)H_{-,+}(n+s)} - \lambda_+$$
$$= \frac{\lambda_+ H_{-,-}(n+s)\Delta_s H_{+,+}(n) + H_{+,-}(n+s)(\lambda_+ H_{-,+}(n+s) - \lambda_- H_{-,+}(n))}{H_{+,+}(n+s)H_{-,-}(n+s) - H_{+,-}(n+s)H_{-,+}(n+s)}.$$

Then, by a standard manipulation of fractions, the first term of the numerator is given by

$$\frac{\lambda_+ H_{-,-}(n+s)\Delta_s H_{+,+}(n)}{H_{+,+}(n+s)H_{-,-}(n+s) - H_{+,-}(n+s)H_{-,+}(n+s)}$$
$$= \frac{\lambda_+ \Delta_s H_{+,+}(n)}{H_{+,+}(n+s)} + \frac{\lambda_+ \Delta_s H_{+,+}(n)}{H_{+,+}(n+s)} \frac{H_{+,-}(n+s)H_{-,+}(n+s)}{(H_{+,+}(n+s)H_{-,-}(n+s) - H_{+,-}(n+s)H_{-,+}(n+s))}.$$

Then, using the bound on the determinant and Assumptions 4.1.1.1, 4.1.1.2 and 4.1.1.6, we have, for all $s \in \mathcal{S}_{a,b}$ and sufficiently large $n \in \mathbb{N}$,

$$\frac{\lambda_+ H_{-,-}(n+s)\Delta_s H_{+,+}(n)}{H_{+,+}(n+s)H_{-,-}(n+s) - H_{+,-}(n+s)H_{-,+}(n+s)} = \frac{\lambda_+ \Delta_s H_{+,+}(n)}{H_{+,+}(n+s)} + \omega\left(\bar{\lambda} e^{-2un}\right).$$

For large $n \in \mathbb{N}$ and all $s \in \mathcal{S}_{a,b}$, by using (4.1.6), the remaining terms satisfy the bound

$$\left| \frac{-\lambda_- H_{+,-}(n+s)H_{-,+}(n) + \lambda_+ H_{+,-}(n+s)H_{-,+}(n+s)}{H_{+,+}(n+s)H_{-,-}(n+s) - H_{+,-}(n+s)H_{-,+}(n+s)} \right|$$
$$\leq \left| \frac{2\bar{\lambda} H_{+,-}(n+s)\left(H_{-,+}(n) + H_{-,+}(n+s)\right)}{H_{+,+}(n+s)H_{-,-}(n+s)} \right|,$$

then, by Assumptions 4.1.1.2, for sufficiently large $n \in \mathbb{N}$ and for all $s \in \mathcal{S}_{a,b}$,

$$\left| \frac{-\lambda_- H_{+,-}(n+s)H_{-,+}(n) + \lambda_+ H_{+,-}(n+s)H_{-,+}(n+s)}{H_{+,+}(n+s)H_{-,-}(n+s) - H_{+,-}(n+s)H_{-,+}(n+s)} \right| \leq C_2 \bar{\lambda} e^{-2un},$$

for some $C_2 > 0$. Combining these calculations we have, for all $s \in \mathcal{S}_{a,b}$ and sufficiently large $n \in \mathbb{N}$,

$$L^{(n)}(s; \{\lambda_\alpha\}_{\alpha \in E}) = \operatorname{diag}\left(\left\{\lambda_\alpha \left(1 + \frac{\Delta_s H_{\alpha,\alpha}(n)}{H_{\alpha,\alpha}(n+s)}\right)\right\}_{\alpha \in E}\right) + C_2 \bar{\lambda} e^{-un}\Omega,$$

where Ω denotes a matrix in which every entry has magnitude less than 1. Hence,

$$L_n(s; \{\lambda_\alpha\}_{\alpha \in E}) = L_{n-1}\left(s; \left\{\lambda_\alpha \left(1 + \frac{\Delta_s H_{\alpha,\alpha}(n)}{H_{\alpha,\alpha}(n+s)}\right)\right\}_{\alpha \in E}\right)$$
$$+ C\bar{\lambda} e^{-un} \left(\prod_{k=0}^{n-1} H(k+s)^{-1}\right) \cdot \Omega \cdot \left(\prod_{k=n-1}^{0} H(k)\right),$$

where $C := \max(6C_1, C_2)$. $\square$

The next lemma relates the norms of H to the diagonal entries and will be useful for proving convergence.

Lemma 4.1.2

Suppose that H satisfies Assumption 4.1.1. Then, for all $\epsilon > 0$, there exists $K \in \mathbb{N}$, such that, for all $k > K$ and $s \in \mathcal{S}_{a,b}$,

$$\|H(k)\| \leq (1+\epsilon)\bar{H}(k) \quad \text{and} \quad \|H(k+s)^{-1}\| \leq (1+\epsilon)\frac{1}{\underline{H}(k)},$$

where $\bar{H}(k) \coloneqq \max_{\alpha \in E} H_{\alpha,\alpha}(k)$, $\underline{H}(k) \coloneqq \min_{\alpha \in E} H_{\alpha,\alpha}(k)$ and $\|\cdot\|$ denotes the norm on $\mathbb{C}^{2\times 2}$ induced by the l^1 norm on $\mathbb{C}^2$.

Proof

For $\alpha, \beta, \gamma \in E$ with $\alpha \neq \beta$ and for large $k \in \mathbb{N}$, there exists a constant $C_1 > 0$, such that

$$|H_{\alpha,\beta}(k)| \leq C_1 e^{-uk} \leq \epsilon |E|^{-1} k^{-1} \leq \epsilon |E|^{-1} |H_{\gamma,\gamma}(k)|,$$

where the first inequality follows from Assumption 4.1.1.1 and last inequality follows from Assumption 4.1.1.7. Then, by considering the operator norm induced by the l^1 norm on $\mathbb{C}^2$,

$$\|H(k)\| = \max_{\beta \in E} \sum_{\alpha \in E} |H_{\alpha,\beta}(k)| \leq (1+\epsilon) \max_{\alpha \in \{+,-\}} |H_{\alpha,\alpha}(k)|.$$

Now consider the inverse. For any $\delta > 0$, for large enough $k \in \mathbb{N}$, by Assumption 4.1.1.2, for all $\alpha, \beta, \gamma \in E$ with $\alpha \neq \beta$ and $s \in \mathcal{S}_{a,b}$, we have that $|H_{\alpha,\beta}(k+s)/H_{\gamma,\gamma}(k+s)| \leq \delta$ and thus, $|\det(H(k+s))| \geq (1-\delta^2)|H_{++}(k+s)H_{--}(k+s)|$. Hence, for all $\alpha \in E$, we have that

$$\left|\left(H(k+s)^{-1}\right)_{\alpha,\alpha}\right| = \frac{|H_{-\alpha,-\alpha}(k+s)|}{|\det(H(k+s))|} \leq \frac{1}{(1-\delta^2)|H_{\alpha,\alpha}(k+s)|}.$$

Moreover, by Assumption 4.1.1.6, for large enough k and all $s \in \mathcal{S}_{a,b}$, we have $|H_{\alpha,\alpha}(k+s)| \leq (1+\epsilon)^{1/2} |H_{\alpha,\alpha}(k)|$. Choose $\delta > 0$ such that $(1-\delta^2)^{-1} \leq (1+\epsilon)^{1/2}$. Then, for all $\alpha \in E$,

$$\left|\left(H(k+s)^{-1}\right)_{\alpha,\alpha}\right| \leq (1+\epsilon)\frac{1}{|H_{\alpha,\alpha}(k)|}.$$

For sufficiently large $k \in \mathbb{N}$, all $\alpha, \beta, \gamma \in \{+,-\}$ with $\alpha \neq \beta$ and $s \in \mathcal{S}_{a,b}$,

$$\begin{aligned}\left|\left(H(k+s)^{-1}\right)_{\alpha,\beta}\right| &= \left|\frac{-H_{\alpha,\beta}(k+s)}{\det(H(k+s))}\right| \\ &\leq \left|\frac{H_{\alpha,\beta}(k+s)}{(1-\delta^2)H_{++}(k+s)H_{--}(k+s)}\right| \leq \frac{C}{1-\delta^2} e^{-uk} \frac{1}{|H_{\gamma,\gamma}(k)|},\end{aligned}$$

where the final inequality follows from Assumption 4.1.1.2. Thus, for sufficiently large $k \in \mathbb{N}$ and all $s \in \mathcal{S}_{a,b}$,

$$\|H(k+s)^{-1}\| \leq (1+\epsilon) \max_{\alpha \in E} \frac{1}{|H_{\alpha,\alpha}(k)|}.$$

$\square$

The following lemma reinforces Assumption 4.1.1.8 by providing a bound which holds for all $n \in \mathbb{N}$ and removing the compensating term for the expression. The cost of doing this is an additional polynomial term within the bound.

Lemma 4.1.3

Suppose Assumption 4.1.1 is satisfied. Then, there exists a $C > 0$ such that, for all $\alpha \in E$, $n \in \mathbb{N}$ and $s \in \mathcal{S}_{a,b}$,

$$\frac{1}{Cn^{b\eta} \exp\left(t_1 \pi |\Im(s)|\right)} \leq \prod_{k=1}^{n} \left(1 + \frac{\Delta_s H_{\alpha,\alpha}(k)}{H_{\alpha,\alpha}(k+s)}\right) \leq Cn^b \exp\left(t_2 \pi |\Im(s)|\right).$$

Proof

From Assumption 4.1.1.8, there is an $N_1 \in \mathbb{N}$ and $C_1, C_2 > 0$ such that, for all $n > N_1$ and $s \in \mathcal{S}_{a,b}$,

$$\left|H_{\alpha,\alpha}(n)^{-s}\right| C_1 \exp\left(-t_1 \pi |\Im(s)|\right) \leq \left|\prod_{k=1}^{n} \left(1 + \frac{\Delta_s H_{\alpha,\alpha}(k)}{H_{\alpha,\alpha}(k+s)}\right)\right| \leq \left|H_{\alpha,\alpha}(n)^{-s}\right| C_2 \exp\left(t_2 \pi |\Im(s)|\right).$$

However, by Assumption 4.1.1.7 and since $s \in \mathcal{S}_{a,b} \subset \mathbb{C}^+$, there exists $C_4, C_5 > 0$ such that, for all $n > N_1$, we have

$$C_4 n^{-b\eta} \exp\left(-t_1 \pi |\Im(s)|\right) \leq \left|\prod_{k=1}^{n} \left(1 + \frac{\Delta_s H_{\alpha,\alpha}(k)}{H_{\alpha,\alpha}(k+s)}\right)\right| \leq C_5 n^b \exp\left(t_2 \pi |\Im(s)|\right).$$

Now, consider $n < N_1$. By Assumption 4.1.1.7, there exists $C_6 < 1 < C_7$, such that, for all $s \in \mathcal{S}_{a,b}$,

$$C_6 k^{-1} |k+s|^{-\eta} \leq \left|\frac{H_{\alpha,\alpha}(k)}{H_{\alpha,\alpha}(k+s)} \leq \right| C_7 k^{\eta} |k+s|.$$

Then, for all $s \in \mathcal{S}_{a,b} \subset \mathbb{C}^+$, we have $|k+s| \leq k + |\Re(s)| + |\Im(s)|$ and $|k+s| \geq k$. Thus,

$$C_6 k^{-1} (k+b+|\Im(s)|)^{-\eta} \leq \sup_{s \in \mathcal{S}_{a,b}} \left|\frac{H_{\alpha,\alpha}(k)}{H_{\alpha,\alpha}(k+s)}\right| \leq C_7 k^{\eta} (k+b+|\Im(s)|).$$

Since we are only considering finitely many values of $k \in \mathbb{N}$, there exist constants $C_8, C_9 > 0$ such that, for all $n < N_1$ and $s \in \mathcal{S}_{a,b}$,

$$C_8 \exp(-t_1 \pi |\Im(s)|) \leq \left|\prod_{k=1}^{n} \left(1 + \frac{\Delta_s H_{\alpha,\alpha}(k)}{H_{\alpha,\alpha}(k+s)}\right)\right| \leq C_9 \exp(t_2 \pi |\Im(s)|).$$

Hence, the result holds for all $n \in \mathbb{N}$ and $s \in \mathcal{S}_{a,b}$. $\square$

We can now consider bounds for L_n by making use of the iterative formula of Lemma 4.1.1.

Lemma 4.1.4

There exists $C > 0$, such that for all $n \in \mathbb{N}$, $s \in \mathcal{S}_{a,b}$ and $\lambda_+, \lambda_- \in \mathbb{C}$,

$$\|L_n(s; \{\lambda_\alpha\}_{\alpha \in E})\| \leq C \max_{\alpha \in E} \left(\lambda_\alpha \prod_{k=1}^{n} \left(1 + \frac{\Delta_s H_{\alpha,\alpha}(k)}{H_{\alpha,\alpha}(k+s)} \right) \right) \exp(t_1 \pi |\Im(s)|),$$
$$\|L_n(s; \{\lambda_\alpha\}_{\alpha \in E})^{-1}\| \leq C \max_{\alpha \in E} \left(\lambda_\alpha^{-1} \prod_{k=1}^{n} \left(1 + \frac{\Delta_s H_{\alpha,\alpha}(k)}{H_{\alpha,\alpha}(k+s)} \right)^{-1} \right) \exp(t_2 \pi |\Im(s)|),$$

where $t_1, t_2 > 0$ are the constants from Assumption 4.1.1.8.

Proof

Fix $s \in \mathcal{S}_{a,b}$ and consider the first inequality. Then, by applying Lemma 4.1.1 repeatedly and taking norms, we have, for some constant $C_1 > 0$,

$$\begin{aligned} \|L_n(s; \{\lambda_\alpha\}_{\alpha \in E})\| \leq{}& \left\| L_0 \left(s; \left\{ \lambda_\alpha \prod_{k=1}^{n} \left(1 + \frac{\Delta_s H_{\alpha,\alpha}(k)}{H_{\alpha,\alpha}(k+s)} \right) \right\}_{\alpha \in E} \right) \right\| \\ &+ \sum_{m=1}^{n} C_1 \|\Omega\| e^{-um} \max_{\alpha \in E} \left(\lambda_\alpha \prod_{k=m+1}^{n} \left(1 + \frac{\Delta_s H_{\alpha,\alpha}(k)}{H_{\alpha,\alpha}(k+s)} \right) \right) \prod_{k=0}^{m-1} \|H(k+s)^{-1}\| \|H(k)\|, \end{aligned}$$

where it is understood that $\prod_{n+1}^{n} x_n := 1$. The L_0 term can be included in the sum, by making a sufficiently large choice of C_1 and changing the starting index to $m = 0$. For all $\epsilon > 0$, there exists an $N \in \mathbb{N}$, such that, for all $k > N$ and $s \in \mathcal{S}_{a,b}$,

$$\|H(k+s)^{-1}\| \|H(k)\| \leq \frac{(1+\epsilon)\bar{H}(k)}{(1-\epsilon)\underline{H}(k)} \leq \frac{(1+\epsilon)^2}{1-\epsilon} e^{v},$$

where the first inequality follows from Lemma 4.1.2 and the second inequality follows from Assumption 4.1.1.3. If we let $v_\epsilon := v \log((1+\epsilon)^2/(1-\epsilon))$, then $\|H(k+s)^{-1}\| \|H(k)\| \leq e^{v_\epsilon}$. Also, define the constant

$$C_2 := \sup_{s \in \mathcal{S}_{a,b}} \max_{n \leq N} \left\{ \prod_{k=0}^{n} \left(\|H(k+s)^{-1}\| \|H(k)\| \right) \right\}, \tag{4.1.7}$$

which is finite by Assumption 4.1.1.10. Then, for all $m \in \mathbb{N}$, we have

$$\prod_{k=0}^{m-1} \left(\|H(k+s)^{-1}\| \|H(k)\| \right) \leq C_2 \exp(\max(m-N, 0) v_\epsilon) \leq C_2 e^{m v_\epsilon}. \tag{4.1.8}$$

Substituting this into the bound for $\|L_n\|$ gives

$$\left\|L_n\left(s;\{\lambda_\alpha\}_{\alpha\in E}\right)\right\| \le C_1C_2\|\Omega\|\sum_{m=0}^{n} e^{(v_\epsilon-u)m}\max_{\alpha\in E}\left(\lambda_\alpha\prod_{k=m+1}^{n}\left(1+\frac{\Delta_s H_{\alpha,\alpha}(k)}{H_{\alpha,\alpha}(k+s)}\right)\right).$$

However, by Lemma 4.1.3,

$$\left|\prod_{k=m+1}^{n}\left(1+\frac{\Delta_s H_{\alpha,\alpha}(k)}{H_{\alpha,\alpha}(k+s)}\right)\right| \le C\left|\prod_{k=1}^{n}\left(1+\frac{\Delta_s H_{\alpha,\alpha}(k)}{H_{\alpha,\alpha}(k+s)}\right)\right| m^{b\eta}\exp(t_1\pi|\Im(s)|),$$

hence,

$$\begin{aligned}\|L_n(s;\{\lambda_\alpha\}_{\alpha\in E})\| &\le C_1C_2\|\Omega\|\max_{\alpha\in E}\left(\lambda_\alpha\prod_{k=1}^{n}\left(1+\frac{\Delta_s H_{\alpha,\alpha}(k)}{H_{\alpha,\alpha}(k+s)}\right)\right)\sum_{m=0}^{n} e^{(v_\epsilon-u)m}m^{b\eta}\exp(t_1\pi|\Im(s)|)\\ &\le C_3\max_{\alpha\in E}\left(\lambda_\alpha\prod_{k=1}^{n}\left(1+\frac{\Delta_s H_{\alpha,\alpha}(k)}{H_{\alpha,\alpha}(k+s)}\right)\right)\exp(t_1\pi|\Im(s)|),\end{aligned}$$

for some constant $C_3>0$, by the Assumption 4.1.1.9.

The bound for $\left\|L_n^{-1}\right\|$ follows from a similar calculation. □

The next lemma gives the convergence result of Theorem 4.1.1. We will make use of the following notation: for $\alpha\in E$, $s\in\mathcal{S}_{a,b}$ and $N,M\in\mathbb{N}$ with $M\le N$, define

$$\pi_\alpha(s,M,N) := \prod_{k=M}^{N}\left(1+\frac{\Delta_s H_{\alpha,\alpha}(k)}{H_{\alpha,\alpha}(k+s)}\right).$$

Lemma 4.1.5

For $s\in\mathcal{S}_{a,b}$, the limit $M(s) := \lim_{n\to\infty} M_n(s)$ exists and is finite. Moreover, for all $T>0$, the convergence is uniform in $\{s\in\mathcal{S}_{a,b}\ :\ |\Im(s)|<T\}$.

Proof

Fix $T>0$. From the definition of M_n and by applying Lemma 4.1.1 repeatedly, there is a constant $C_1>0$ such that, for all $s\in\mathcal{S}_{a,b}$ and $n,m\in\mathbb{N}$ with $m<n$,

$$\begin{aligned}&M_n(s) = L_m\left(s;\{H_{\alpha,\alpha}(n)^s\pi_\alpha(s,m+1,n)\}_{\alpha\in E}\right)\\ &+\sum_{k=m+1}^{n} C_1\max_{\alpha\in E}\left(|H_{\alpha,\alpha}(n)^s|\pi_\alpha(s,k+1,n)\right)e^{-uk}\left(\prod_{l=0}^{k-1}H(l+s)^{-1}\right)\cdot\Omega\cdot\left(\prod_{l=k-1}^{0}H(l)\right).\end{aligned}$$

The first term of the above expression is given by

$$\begin{aligned}&L_m\left(s;\{H_{\alpha,\alpha}(n)^s\pi_\alpha(s,m+1,n)\}_{\alpha\in E}\right)\\ &\qquad= L_m\left(s;\{H_{\alpha,\alpha}(n)^s\pi_\alpha(s,m+1,n)+H_{\alpha,\alpha}(m)^s-H_{\alpha,\alpha}(m)^s\}_{\alpha\in E}\right)\\ &\qquad= M_m(s)+L_m\left(s;\{H_{\alpha,\alpha}(n)^s\pi_\alpha(s,m+1,n)-H_{\alpha,\alpha}(m)^s\}_{\alpha\in E}\right).\end{aligned}$$

Then, by applying Lemma 4.1.4, there is a constant $C_2 > 0$ such that

$$\begin{aligned}
&\|L_m(s; \{H_{\alpha,\alpha}(n)^s \pi_\alpha(s, m+1, n) - H_{\alpha,\alpha}(m)^s\}_{\alpha \in E})\| \\
&\qquad \leq C_2 \exp\left(t_1 \pi |\Im(s)|\right) \max_{\alpha \in E} \left\{\left(H_{\alpha,\alpha}(n)^s \pi_\alpha(s, m+1, n) - H_{\alpha,\alpha}(m)^s\right) \pi_\alpha(s, 1, m)\right\} \\
&\qquad = C_2 \exp\left(t_1 \pi |\Im(s)|\right) \max_{\alpha \in E} \left\{H_{\alpha,\alpha}(n)^s \pi_\alpha(s, 1, n) - H_{\alpha,\alpha}(m)^s \pi_\alpha(s, 1, m)\right\}.
\end{aligned}$$

However, by Assumption 4.1.1.8, $H_{\alpha,\alpha}(n)^s \pi_\alpha(s, 1, n)$ is uniformly Cauchy in $\{s \in \mathcal{S}_{a,b} : |\Im(s)| < T\}$. Hence, for any $\epsilon > 0$, there exists $N_1 \in \mathbb{N}$ such that, whenever $n, m > N_1$ and $s \in \mathcal{S}_{a,b}$ with $|\Im(s)| < T$, we have

$$\max_{\alpha \in E} \left\{\left(H_{\alpha,\alpha}(n)^s \pi_\alpha(s, 1, n) - H_{\alpha,\alpha}(m)^s\right) \pi_\alpha(s, 1, m)\right\} < \epsilon,$$

and so

$$\|L_m(s; \{H_{\alpha,\alpha}(n)^s \pi_\alpha(s, m+1, n) - H_{\alpha,\alpha}(m)^s\}_{\alpha \in E})\| \leq \epsilon C_2 \exp\left(t_1 \pi |\Im(s)|\right).$$

Substituting this into the expression for $M_n(s)$, taking norms and using the inequality (4.1.7) gives

$$\|M_n(s) - M_m(s)\| \leq \epsilon C_2 \exp(t_1 \pi |\Im(s)|) + C_3 \sum_{k=m+1}^{n} \max_{\alpha \in \{+,-\}} \left(|H_{\alpha,\alpha}(n)^s| \pi_\alpha(s, k+1, n)\right) e^{(v_\epsilon - u)k},$$

where C_3 is the constant from (4.1.8). Then, for each $\alpha \in E$, using Lemma 4.1.3, we have, for some constant $C_4 > 0$,

$$\begin{aligned}
|H_{\alpha,\alpha}(n)^s \pi_\alpha(s, k+1, n)| &= |H_{\alpha,\alpha}(n)^s \pi_\alpha(s, 1, n)| \left(|\pi_\alpha(s, 1, k)|\right)^{-1} \\
&\leq |H_{\alpha,\alpha}(n)^s \pi_\alpha(s, 1, n)| C_4 k^{b\eta} \exp(t_1 \pi |\Im(s)|).
\end{aligned}$$

From Assumption 4.1.1.8, for sufficiently large $n \in \mathbb{N}$, the sequence $H_{\alpha,\alpha}(n)^s \pi_\alpha(s, 1, n)$ is uniformly bounded for $s \in \mathcal{S}_{a,b}$ with $|\Im(s)| < T$. Thus, there is a constant $C_5 > 0$, such that for sufficiently large $m \in \mathbb{N}$,

$$\|M_n(s) - M_m(s)\| \leq \epsilon C_2 \exp(t_1 \pi T) + C_5 \sum_{k=m+1}^{n} k^{b\eta} e^{(v_\epsilon - u)k}.$$

Then, since $v_\epsilon - u < 0$ by Assumption 4.1.1.9, the series $\sum_{k=m+1}^{\infty} k^{b\eta} e^{(v_\epsilon - u)k}$ converges. So by taking m sufficiently large, the sum can be made arbitrarily small. Moreover, notice that it has no dependence on s. Hence, $M_n(s)$ is a uniformly Cauchy sequence and so converges uniformly within the domain $\{s \in \mathcal{S}_{a,b} : |\Im(s)| < T\}$. □

4.1.2 Properties of M

The proofs of this section will make use of the notation ω, which is outline in the Abbreviations and Notation section.

The first property of M which we consider is the recurrence relation (2.4.1) and the extension of M outside of $\mathcal{S}_{a,b}$, in Proposition 4.1.1.

Proof of Proposition 4.1.1
We will prove the result of (i), where $z = s + N$ for $N \in \mathbb{N}$, and note that the proof of (ii), where $z = s - N$, is very similar.

From the definition of M and the substitution $z = s + N$, it follows that

$$\begin{aligned}
&\left(\prod_{k=N-1}^{0} H(s+k)\right) M(s) \\
&\quad = \lim_{n\to\infty} \left(\prod_{k=N-1}^{0} H(s+k)\right) \left(\prod_{k=0}^{n} H(k+s)^{-1}\right) \operatorname{diag}\left(\{H_{\alpha,\alpha}(n)^{s}\}_{\alpha\in E}\right) \left(\prod_{k=n}^{0} H(k)\right) \\
&\quad = \lim_{n\to\infty} \left(\prod_{k=0}^{n-N} H(z+k)^{-1}\right) \operatorname{diag}\left(\left\{H_{\alpha,\alpha}(n)^{z-N}\right\}_{\alpha\in E}\right) \left(\prod_{k=n}^{0} H(k)\right).
\end{aligned}$$

However, for $k \in \{n-N+1, \cdots, n\}$,

$$\begin{aligned}
&\operatorname{diag}\left(\left\{\frac{H_{\alpha,\alpha}(n)\cdots H_{\alpha,\alpha}(k+1)}{H_{\alpha,\alpha}(n)^{N}}\right\}_{\alpha\in E}\right) H(k) \\
&\quad = \operatorname{diag}\left(\left\{\frac{H_{\alpha,\alpha}(n)\cdots H_{\alpha,\alpha}(k)}{H_{\alpha,\alpha}(n)^{N}}\right\}_{\alpha\in E}\right) + \omega\left(\max_{\alpha\in E} \frac{H_{\alpha\alpha}(n)\cdots H_{\alpha\alpha}(k+1)}{H_{\alpha\alpha}(n)^{N}} H_{\alpha,-\alpha}(k)\right)\Omega.
\end{aligned}$$

Hence, by induction,

$$\begin{aligned}
\operatorname{diag}\left(\left\{H_{\alpha,\alpha}(n)^{-N}\right\}_{\alpha\in E}\right) \left(\prod_{k=n}^{n-N+1} H(k)\right) &= \operatorname{diag}\left(\left\{\frac{H_{\alpha,\alpha}(n)\cdots H_{\alpha,\alpha}(n-N+1)}{H_{\alpha,\alpha}(n)^{N}}\right\}_{\alpha\in E}\right) \\
&\quad + \sum_{k=n-N+1}^{n} \omega\left(\max_{\alpha\in E} \frac{H_{\alpha,\alpha}(n)\cdots H_{\alpha,\alpha}(k+1)}{H_{\alpha,\alpha}(n)^{N}} H_{\alpha,-\alpha}(k)\right) \cdot \Omega \cdot \prod_{l=k-1}^{n-N+1} H(l).
\end{aligned}$$

Then, by applying Assumption 4.1.1.6 and Assumption 4.1.1.2 to the sum and by applying Assumption 4.1.1.6 to the diagonal matrix, we have, for sufficiently large $n \in \mathbb{N}$,

$$\begin{aligned}
&\operatorname{diag}\left(\left\{H_{\alpha,\alpha}(n)^{-N}\right\}_{\alpha\in E}\right) \left(\prod_{k=n}^{n-N+1} H(k)\right) \\
&\quad = \operatorname{diag}\left(\left\{(1+\omega(\epsilon))^{N}\right\}_{\alpha\in E}\right) + \sum_{k=n}^{n-N+1} (1+\epsilon)^{N} \omega(e^{-uk}) \cdot \Omega \cdot \prod_{l=k-1}^{n-N+1} H(l). \qquad (4.1.9)
\end{aligned}$$

Using Lemma 4.1.2 and Assumption 4.1.1.7, we have that $\|H(k)\| \leq C_1 k^\eta$ for sufficiently large $k \in \mathbb{N}$ and some constant $C_1 > 0$. Hence, for large $n \in \mathbb{N}$,

$$\left\| \sum_{k=n}^{n-N+1} (1+\epsilon)^N \omega(e^{-uk}) \Omega \prod_{l=k-1}^{n-N+1} H(l) \right\| \leq e^{-u(n-N)} \|\Omega\| (1+\epsilon)^N \sum_{k=n}^{n-N+1} \prod_{l=k-1}^{n-N+1} C_1 l^\eta \leq C_2 e^{-un},$$

for some constant $C_2 > 0$. Thus, for all $\epsilon > 0$ and sufficiently large $n \in \mathbb{N}$,

$$\operatorname{diag}\left(\left\{ H_{\alpha,\alpha}(n)^{-N} \right\}_{\alpha \in E} \right) \left(\prod_{k=n}^{n-N+1} H(k) \right) = \operatorname{diag}\left(\{1 + \omega(\epsilon)\}_{\alpha \in E} \right) + C_2 e^{-un} \Omega,$$

where $(1+\epsilon)^N$ has been replaced by $(1+\epsilon)$ since $\epsilon > 0$ was arbitrary and $N \in \mathbb{N}$ is fixed. Moreover, for sufficiently large $n \in \mathbb{N}$, $|H_{\alpha,\alpha}(n-N)^z / H_{\alpha,\alpha}(n) - 1| \leq \epsilon$ by Assumption 4.1.1.6. Combining these two inequalities gives

$$\begin{aligned}
&\left(\prod_{k=N-1}^{0} H(k+s) \right) M(s) \\
&\quad = \lim_{n \to \infty} \left(\prod_{k=0}^{n-N} H(k+z)^{-1} \right) \operatorname{diag}\left(\{ H_{\alpha,\alpha}(n-N)^z (1 + \omega(\epsilon)) \}_{\alpha \in E} \right) \left(\prod_{k=n-N}^{0} H(k) \right) \\
&\quad + C_2 e^{-un} \lim_{n \to \infty} \left(\prod_{k=0}^{n-N} H(k+z)^{-1} \right) \operatorname{diag}\left(\{ H_{\alpha,\alpha}(n-N)^z \}_{\alpha \in E} \right) \Omega \left(\prod_{k=n-N}^{0} H(k) \right),
\end{aligned}$$

where the second term converges to 0 by the arguments of Lemma 4.1.4 and the first term is given by $M_{n-N}(z) + L_{n-N}(z; \{H_{\alpha,\alpha}(n-N)^z \omega(\epsilon)\}_{\alpha \in E})$. We now consider the L_{n-N} term.

By reversing the shift of indices, we have

$$\begin{aligned}
&L_{n-N}(z; \{H_{\alpha,\alpha}(n-N)^z \omega(\epsilon)\}_{\alpha \in E}) \\
&= \lim_{n \to \infty} \left(\prod_{k=0}^{n-N} H(k+z)^{-1} \right) \operatorname{diag}\left(\{ H_{\alpha,\alpha}(n-N)^z \omega(\epsilon) \}_{\alpha \in E} \right) \left(\prod_{k=n-N}^{0} H(k) \right) \\
&= \left(\prod_{k=N-1}^{0} H(k+s) \right) \lim_{n \to \infty} \left(\prod_{k=0}^{n} H(k+s)^{-1} \right) \operatorname{diag}\left(\{ H_{\alpha,\alpha}(n-N)^z \omega(\epsilon) \}_{\alpha \in E} \right) \\
&\qquad \times \left(\prod_{k=n-N+1}^{n} H(k)^{-1} \right) \left(\prod_{k=n}^{0} H(k) \right).
\end{aligned}$$

By a similar argument to before, we have that

$$\begin{aligned}
&\operatorname{diag}\left(\{ H_{\alpha,\alpha}(n-N)^z \omega(\epsilon) \}_{\alpha \in E} \right) \left(\prod_{k=n-N+1}^{n} H(k)^{-1} \right) \\
&\qquad\qquad = \operatorname{diag}\left(\left\{ \omega(\epsilon(1+\epsilon)^N) H_{\alpha,\alpha}(n)^{z-N} \right\}_{\alpha \in E} \right) + C_3 \omega(\epsilon) e^{-u(n-N+1)} \Omega,
\end{aligned}$$

for some constant $C_3 > 0$. Thus,

$$
\begin{aligned}
&L_{n-N}\left(z; \{H_{\alpha,\alpha}(n-N)^z \omega(\epsilon)\}_{\alpha\in E}\right) \\
&\qquad = \left(\prod_{k=N-1}^{0} H(k+s)\right) L_n\left(s; \left\{\omega(\epsilon(1+\epsilon)^N) H_{\alpha,\alpha}(n)^{z-N}\right\}_{\alpha\in E}\right) \\
&\qquad\qquad + C_3 \omega(\epsilon) e^{-un} \left(\prod_{k=N}^{n} H(k+s)^{-1}\right) \cdot \Omega \cdot \left(\prod_{k=n}^{0} H(k)\right), \qquad (4.1.10)
\end{aligned}
$$

where the second term converges to zero by standard arguments. Hence, by Lemma 4.1.4, for all $\epsilon > 0$ and sufficiently large n, there exists a constant $C_4 > 0$ such that

$$
\begin{aligned}
&\|L_{n-N}\left(z; \{\omega(\epsilon) H_{\alpha,\alpha}(n-N)^z\}_{\alpha\in E}\right)\| \\
&\leq \epsilon + C_4 \epsilon \left(\prod_{k=0}^{N-1} \|H(k+s)\|\right) \max_{\alpha\in E}\left(H_{\alpha,\alpha}(n)^s \prod_{k=1}^{n}\left(1 + \frac{\Delta_s H_{\alpha,\alpha}(k)}{H_{\alpha,\alpha}(k+s)}\right)\right) \exp(\alpha_1 \pi |\Im(s)|).
\end{aligned}
$$

However, by Assumption 4.1.1.8, the 'max' term is bounded for $s \in \mathcal{S}_{a,b}$, hence $\left\|L_{n-N}\left(z; \{\omega(\epsilon) H_{\alpha,\alpha}(n-N)^z\}_{\alpha\in E}\right)\right\| < C_5 \epsilon$, for some constant $C_5 > 0$. Thus, since $\epsilon > 0$ was arbitrary,

$$
M(z) \coloneqq \lim_{n\to\infty} M_{n-N}(z) = \left(\prod_{k=N-1}^{0} H(k+s)\right) M(s).
$$

Hence, $M(z)$ is well defined and satisfies (4.1.1). □

The next lemma considers when M is an invertible matrix.

Lemma 4.1.6

For each $s \in \mathcal{S}_{a,b}$, $M(s)$ is non-singular.

Proof

For each $s \in \mathcal{S}_{a,b}$, by the definition of $\{M_n\}_{n\in\mathbb{N}}$, for all $n \in \mathbb{N}$,

$$
\det M_n(s) = \left(\prod_{k=0}^{n} \frac{\det H(k)}{\det H(k+s)}\right) H_{++}(n)^s H_{--}(n)^s,
$$

where, for sufficiently large $n \in \mathbb{N}$, $H_{++}(n), H_{--}(n) \neq 0$ by Assumption 4.1.1.7. Moreover, by Assumption 4.1.1.2,

$$
\lim_{n\to\infty} \frac{\det H(n)}{H_{++}(n) H_{--}(n)} = 1,
$$

hence,

$$
\det M(s) = \lim_{n\to\infty} \det M_n(s) = \lim_{n\to\infty} \det H(n)^s \left(\prod_{k=0}^{n} \frac{\det H(k)}{\det H(k+s)}\right) =: \Gamma_{\det H}(s).
$$

That is, $\det M(s)$ is of the form of the *generalised Gamma function* or *Webster product* corresponding to $\det H(s)$ considered in [53].

By Assumption 4.1.1.2, there exists $C_1 > 0$ such that, for all $s \in \mathcal{S}_{a,b} \cup \{0\}$ and $k \in \mathbb{N}$,

$$\begin{aligned}\det H(k+s) &= H_{++}(k+s)H_{--}(k+s)\left(1 - \frac{H_{-+}(k+s)H_{+-}(k+s)}{H_{++}(k+s)H_{--}(k+s)}\right)\\ &= H_{++}(k+s)H_{--}(k+s)\left(1 - C_1\omega(e^{-2u(k+s)})\right).\end{aligned}$$

Then, by taking the logarithm of both sides, we have

$$\log(\det H(k+s)) = \log(H_{++}(k+s)) + \log(H_{--}(k+s)) + \log\left(1 - C_1\omega(e^{-2u(k+s)})\right).$$

However, there exists $N \in \mathbb{N}$, such that $|C_1 e^{-2u(k+s)}| < \frac{1}{2}$, for all $k > N$. Hence, from the Taylor expansion of log, for all $k > N$,

$$\left|\log\left(1 - \omega(e^{-2u(k+s)})\right)\right| = \log(1) + \left|C_1\omega(e^{-2u(k+s)})\frac{1}{1 + C_1\omega(e^{-2u(k+s)})}\right| \leq C_2 e^{-2uk},$$

for some positive constant C_2.

Moreover, for each $k < N$, by Assumption 4.1.1.5 we have that $\det(H(k+s)) \neq 0$ for all $s \in \mathcal{S}_{a,b} \cup \{0\}$ and $k \in \mathbb{N}$. Hence, since N is finite,

$$R(s) := \prod_{k=0}^{N}\left|1 - \frac{H_{-+}(k+s)H_{+-}(k+s)}{H_{++}(k+s)H_{--}(k+s)}\right| = \prod_{k=0}^{N}\left|\frac{\det(H(k+s))}{H_{++}(k+s)H_{--}(k+s)}\right| \neq 0.$$

Then, we have

$$\log(\det(M_n(s))) = \sum_{\alpha\in E}\log\left(H_{\alpha,\alpha}(n)^s \prod_{k=0}^{n}\frac{H_{\alpha,\alpha}(k)}{H_{\alpha,\alpha}(k+s)}\right) + \log(R(s)) + \sum_{k=N+1}^{\infty} C_2\omega(e^{-2uk}).$$

Hence, by computing the bounds on the final sum and making use of Assumption 4.1.1.8,

$$|\det(M(s))| \geq \frac{R(s)}{C_3}\exp\left(-2\alpha_1\pi|\Im(s)| - \frac{C_2}{1 - e^{-2u}}\right) > 0,$$

for some $C_3 > 0$. Thus, $M(s)$ is non-singular. □

Another important property of M which we will need is that it is analytic in certain regions.

Lemma 4.1.7

The entries of M are analytic on $\mathcal{S}_{a,b}$.

Proof

The components of $H(k+s)$ are analytic, as a function of s, for $s \in \mathcal{S}_{a,b}$ and $k \in \mathbb{N}$ by

Assumption 4.1.1.4. By Assumption 4.1.1.5, we have that $\det(H(k+s)) \neq 0$ for all $k \in \mathbb{N}$ and $s \in \mathcal{S}_{a,b}$. Hence $H(k+s)^{-1}$ is analytic as a function of s over $\mathcal{S}_{a,b}$.

For each $T > 0$, by Lemma 4.1.5, $M_n(s)$ converges uniformly to $M(s)$ in $\{s \in \mathcal{S}_{a,b} : |\Im(s)| < T\}$ as $n \to \infty$. Hence, M is an analytic function on $\{s \in \mathcal{S}_{a,b} : |\Im(s)| < T\}$. Then, by considering successively larger values of T, we see that M is analytic on all of $\mathcal{S}_{a,b}$.

□

In the following lemma the bounds on $\|L_n\|$ are used to obtain bounds for $\|M\|$.

Lemma 4.1.8

There are constants $C_1, C_2, C_3 > 0$, such that for all $s \in \mathcal{S}_{a,b}$,

$$\begin{aligned}\|M(s)\| &\leq C_1 \lim_{n\to\infty} \max_{\alpha\in\{+,-\}} \left(H_{\alpha,\alpha}(n)^s \prod_{k=1}^{n} \left(1 + \frac{\Delta_s H_{\alpha,\alpha}(k)}{H_{\alpha,\alpha}(k+s)}\right)\right) \exp\left(t_1 \pi |\Im(s)|\right) \\ &\leq C_2 \exp((t_1 + t_2)\pi |\Im(s)|)\end{aligned}$$

and

$$\|M(s)^{-1}\| \leq C \exp((t_1 + t_2)\pi |\Im(s)|),$$

where $t_1, t_2 > 0$ are the constants from Assumption 4.1.1.8.

Proof

By Lemma 4.1.4, there exists a constant $C_1 > 0$ such that, for all $s \in \mathcal{S}_{a,b}$ and $n \in \mathbb{N}$,

$$\|M_n(s)\| \leq C_1 \max_{\alpha\in\{+,-\}} \left(H_{\alpha,\alpha}(n)^s \prod_{k=1}^{n} \left(1 + \frac{\Delta_s H_{\alpha,\alpha}(k)}{H_{\alpha,\alpha}(k+s)}\right)\right) \exp\left(t_1 \pi |\Im(s)|\right),$$

which is the first result of the lemma. Thus, using Assumption 4.1.1.8 and taking the limit as $n \to \infty$, we obtain the second result.

By applying Lemma 4.1.4, there is also a constant $C_3 > 0$ such that, for all $n \in \mathbb{N}$ and $s \in \mathcal{S}_{a,b}$,

$$\left\| L_n\left(s, \{H_{\alpha,\alpha}(n)^s\}_{\alpha\in E}\right)^{-1} \right\| \leq C_3 \max_{\alpha\in E} \left(H_{\alpha,\alpha}(n)^{-s} \prod_{k=1}^{n} \left(1 + \frac{\Delta_s H_{\alpha,\alpha}(k)}{H_{\alpha,\alpha}(k+s)}\right)^{-1}\right) \exp\left(t_2 \pi |\Im(s)|\right).$$

Hence, by Assumption 4.1.1.8, for some constant $C_4 > 0$, there exists $N \in \mathbb{N}$ such that, for all $n > N$, we have

$$\left\| L_n\left(s, \{H_{\alpha,\alpha}(n)^s\}_{\alpha\in E}\right)^{-1} \right\| \leq C_4 \exp\left((t_1 + t_2)\pi |\Im(s)|\right),$$

from which the final result of the lemma is obtained by taking the limit as $n \to \infty$. $\square$

We are now able to prove Theorem 4.1.1.

Proof of Theorem 4.1.1

Suppose $s \in \mathcal{S}_{a,b}$. The existence of the limit $M(s) := \lim_{n\to\infty} M_n(s)$ is given by Lemma 4.1.5. The recurrence relation follows from Proposition 4.1.1 with $N = 1$. The upper bounds of $\|M(s)^{-1}\|$ and $\|M(s)\|$ follow from Lemma 4.1.8. Finally, by Lemma 4.1.7 we have that the entries of M are analytic in the strip $\mathcal{S}_{a,b}$.

$\square$

4.1.3 Mellin Transform

We now consider the proof of Theorem 4.1.2. Suppose $a, b \in \mathbb{R}$ and $M : \mathbb{C} \to \mathbb{C}^{2\times 2}$ are such that the conditions of Theorem 4.1.2 hold. Define the functions $\mathbf{h}, \mathbf{f} : \mathbb{C}^+ \to \mathbb{C}^2$ by $\mathbf{h} := M(s)\mathbf{m}$ and $\mathbf{f} := \mathbf{E}[A_\infty^s]$, where $\mathbf{m} \in \mathbb{C}^2$ is as defined in Theorem 4.1.2. From the properties of M, we have the following bounds on $\mathbf{h}$.

Lemma 4.1.9

Under the assumptions of Theorem 4.1.2, there exist constants $C_3, C_4 > 0$, such that for all $s \in \mathcal{S}_{a,b}$

$$C_3 \exp(-t_2\pi|\Im(s)|) \leq \|\mathbf{h}(s)\| \leq C_4 \exp(t_1\pi|\Im(s)|).$$

Proof

Using the bounds on $\|M(s)\|$ assumed in Theorem 4.1.2, for all $s \in \mathcal{S}_{a,b}$,

$$\|\mathbf{h}(s)\| = \|M(s)\mathbf{m}\| \leq \|M(s)\|\|\mathbf{m}\| \leq \|\mathbf{m}\|C_1 \exp\left(t_1\pi|\Im(s)|\right).$$

For the lower bound, by the assumptions of Theorem 4.1.2,

$$\|\mathbf{m}\| = \|M^{-1}(s)\mathbf{h}(s)\| \leq \|M^{-1}(s)\|\|\mathbf{h}(s)\| \leq C_2 \exp(t_2\pi|\Im(s)|)\|\mathbf{h}(s)\|,$$

which can be rearranged to give

$$\|\mathbf{h}(s)\| \geq \|\mathbf{m}\|C_2^{-1} \exp(-t_2\pi|\Im(s)|).$$

$\square$

Define the function $T : \mathcal{S}_{a,b} \to \mathbb{C}$ by $T(s) := f_+(s)h_-(s) - f_-(s)h_+(s)$. The following lemma is adapted from the proof of [35, pp 13, Section 3, Theorem 3.9], where the changes are due to the fact that in this setting there isn't an analytic expression for the Mellin transform.

Lemma 4.1.10

Under the assumptions of Theorem 4.1.2, $T(s) = 0$, for all $s \in \mathcal{S}_{a,b}$.

Proof

For each $\alpha \in E$, the function f_α satisfies the bound

$$|f_\alpha(s)| = |\mathbb{E}_\alpha[A_\infty^s]| \leq \mathbb{E}_\alpha[|A_\infty^s|] \leq \mathbb{E}_\alpha\left[A_\infty^{\Re(s)}\right], \tag{4.1.11}$$

for all $s \in \mathcal{S}_{a,b}$. However, $\mathbb{E}_\alpha\left[A_\infty^{\Re(s)}\right] < \infty$ by the assumptions of Theorem 4.1.2. Then, since $\mathbb{E}_\alpha\left[A_\infty^{\Re(s)}\right]$ is continuous, it is bounded over the closed interval $[a, b]$ and thus f_α is bounded over $\mathcal{S}_{a,b}$.

Notice that, $T(s)$ is given by the determinant

$$T(s) = \det\begin{pmatrix} f_+(s) & h_+(s) \\ f_-(s) & h_-(s) \end{pmatrix}.$$

Moreover, by the recurrence relation (4.0.1) on the moments of A_∞ and the assumption (4.1.5) on M, we have, for $s + 1 \in \mathcal{S}_{a,b}$,

$$\begin{pmatrix} f_+(s+1) & h_+(s+1) \\ f_-(s+1) & h_-(s+1) \end{pmatrix} = \left(\frac{-F(s+1)}{s+1}\right)^{-1}\begin{pmatrix} f_+(s) & h_+(s) \\ f_-(s) & h_-(s) \end{pmatrix},$$

hence, T satisfies the recurrence relation

$$T(s+1) = \det\left(\frac{-F(s+1)}{s+1}\right)^{-1} T(s). \tag{4.1.12}$$

It follows from (4.1.5) that $\det(M(s))$ also satisfies this relation. By *Hadamard's inequality* for 2×2 matrices (see Appendix C.1) and our choice of norms, there exists $C_1 > 0$ such that $|\det(M(s))| \geq C_1\|M(s)^{-1}\|^{-2}$. By assumption (5) of Theorem 4.1.2, there exists $C_2 > 0$ such that $\|(M(s))^{-1}\| \leq C_2 \exp(t_2\pi|\Im(s)|)$ for any $s \in \mathcal{S}_{a,b}$, hence,

$$|\det(M(s))| \geq C_3 \exp(-2t_2\pi|\Im(s)|),$$

for some $C_3 > 0$. Assumption (2) of Theorem 4.1.2, implies that $\det(M(s))$ is zero-free in $\mathcal{S}_{a,b}$. Moreover, by assumption (3) of Theorem 4.1.2, the entries of M are analytic, hence

$\det(M)$ is also analytic. Then, since f_+, f_-, h_+ and h_- are analytic in $\mathcal{S}_{a,b}$, so must be T. Hence, the function $H : \mathcal{S}_{a,b} \to \mathbb{C}$, defined by

$$H(s) := \frac{T(s)}{\det(M(s))},$$

is analytic over $\mathcal{S}_{a,b}$. Moreover, since both T and $\det(M)$ satisfy the recurrence relation (4.1.5), $H(s+1) = H(s)$ whenever $s, s+1 \in \mathcal{S}_{a,b}$. Thus, we can extend H to an entire periodic function, with period 1.

The bounds on h_+, h_-, f_+ and f_- from Lemma 4.1.9 and (4.1.11) imply that there is a constant $C_4 > 0$ such that $|T(s)| \leq C_4 \exp(t_1\pi|\Im(s)|)$, for all $s \in \mathcal{S}_{a,b}$. Then, for each $s \in \mathcal{S}_{a,b}$,

$$|H(s)| \leq \frac{C_4 \exp(t_1\pi|\Im(s)|)}{C_3 \exp(-2t_2\pi|\Im(s)|)} \leq C_5 \exp\left((t_1 + 2t_2)\pi|\Im(s)|\right),$$

where $C_5 > 0$ is some constant and $2t_2 + t_1 \in (0, 2)$ by assumption (1) of Theorem 4.1.2. Since H is periodic, this bound can be extended to all $s \in \mathbb{C}$.

Hence, H is an analytic period function in the entire complex plane and satisfies the bound $|H(s)| \leq C\exp(\gamma\pi|\Im(s)|)$ for all $s \in \mathbb{C}$, for some constants $C > 0$ and $\gamma := t_1 + 2t_2 \in (0, 2)$. Within the proof of [34, pp 11, Proposition 2], it is shown that these conditions are sufficient for H to be constant (see Appendix C.2).

Then, since $b > a + 1$, there exists an $n \in \mathbb{Z}$ such that $n \in \mathcal{S}_{a,b}$. By inductively applying the recurrence relation for the moments of A_∞, we have $f_+(n) = h_+(n)$ and $f_-(n) = h_-(n)$. Then, it is immediate that $T(n) = 0$, hence $H(n) = 0$. Since H is constant, this means $H \equiv 0$ and, by definition, this can only be true if $T \equiv 0$.

$\square$

We are now in a position to prove Theorem 4.1.2.

Proof of Theorem 4.1.2

Lemma 4.1.10 implies that $0 = f_+(s)h_-(s) - f_-(s)h_+(s)$, for all $s \in \mathcal{S}_{a,b}$. Since $M(s)$ is non-singular by assumption (2) of Theorem 4.1.2, at least one of $h_-(s)$ and $h_+(s)$ is non-zero. Thus, for each $\alpha \in E$, $h_\alpha(s) = 0$ if and only if $f_\alpha(s) = 0$. Therefore, we define, for all $s \in \mathcal{S}_{a,b}$,

$$w(s) := \begin{cases} \frac{f_+(s)}{h_+(s)}, & \text{if } h_+(s) \neq 0; \\ \frac{f_-(s)}{h_-(s)}, & \text{otherwise.} \end{cases}$$

Then, notice that $f_\alpha(s) = w(s)h_\alpha(s)$ for each $\alpha \in E$ and, whenever $h_+(s)$ and $h_-(s)$ are both non-zero, $w(s) = \frac{f_+(w)}{h_+(s)} = \frac{f_-(s)}{h_-(s)}$.

For all $s \in \mathcal{S}_{a,b}$, there is an $\alpha \in E$ such that $h_\alpha(s) \neq 0$. Then, since h_α is continuous, there is a neighbourhood around s in which h_α is zero free. Since f_α and h_α are both analytic, w is also analytic in that neighbourhood. This holds for all $s \in \mathcal{S}_{a,b}$, hence w is analytic in $\mathcal{S}_{a,b}$.

For $\alpha \in E$, by the recurrence relation (4.0.1), which holds for both $\mathbf{f}$ and $\mathbf{h}$, whenever $s, s+1 \in \mathcal{S}_{a,b}$,

$$\begin{aligned} w(s+1)\mathbf{h}(s+1) = \mathbf{f}(s+1) = \left(\frac{-F(s+1)}{s+1}\right)^{-1} \mathbf{f}(s) \\ = w(s)\left(\frac{-F(s+1)}{s+1}\right)^{-1} \mathbf{h}(s) = w(s)\mathbf{h}(s+1). \end{aligned}$$

Thus, $w(s+1) = w(s)$ and so w can be extended to an entire periodic function on $\mathbb{C}$, with period 1.

From (4.1.11), we know that f_α is bounded in $\mathcal{S}_{a,b}$ for each $\alpha \in \{+,-\}$, whilst, by Lemma 4.1.9, for some $C_3 > 0$ we have that $\max(|h_+(s)|, |h_-(s)|) \geq C_3 \exp(-t_2\pi|\Im(s)|)$, for all $s \in \mathcal{S}_{a,b}$. Thus, $w(s) \leq C_4 \exp(t_2\pi|\Im(s)|)$ for all $s \in \mathcal{S}_{a,b}$, for some constant $C_4 > 0$. Since w is periodic with period 1 and $\mathcal{S}_{a,b}$ is a strip of width greater than 1, this bound can be extended to all $s \in \mathbb{C}$. Then, following the argument from the proof of [34, pp 11, Proposition 2], w is constant (see Appendix C.2 for details).

Since $b > a+1$, there exists $n \in \mathbb{Z}$ such that $n \in \mathcal{S}_{a,b}$. Then, $\mathbf{f}(n) = \mathbf{h}(n)$, by (4.1.3) and (2.4.2) if $n > 0$ or by (4.1.4) and (2.4.3) if $n < 0$. We can therefore conclude $w \equiv 1$. Then, it is immediate that $\mathbf{f}(s) = \mathbf{h}(s)$ for all $s \in \mathcal{S}_{a,b}$.

Moreover, since M is defined for all $s \in D^*$ and continues to satisfy the recurrence relation (4.0.1), we have that $\mathbf{E}[A_\infty^s] = M(s)\mathbf{m}$ for all $s \in D^*$. Since $\mathbf{E}[A_\infty^s]$ is an analytic function of s in D, the function M must have an analytic continuation to D. □

4.2 Specific Results

Consider the MAP $\{(J_t, \xi_t) : t \geq 0\}$ and let the state space of J be given by $E := \{+,-\}$. The two theorems of Section 4.1 can be used to represent the Mellin transform of the density of A_∞ in certain cases. We will prove results for two such cases here.

For ease of notation, for each $\alpha \in E$, $s \in \mathbb{C}$ and $n \in \mathbb{N}$, let $K_\alpha(s) := \frac{q_\alpha - \psi_\alpha(s)}{s}$ and $K_\alpha := \lim_{x\to\infty} K_\alpha(s)$.

4.2.1 Non Increasing MAP with Zero Drift

Suppose the MAP $\{(J_t, \xi_t) : t \geq 0\}$ satisfies the following assumptions.

Assumption 4.2.1

The process (J, ξ) is a MAP and $a, b \in \mathbb{R}$ such that:

1. *The process $\{\xi_t : t \geq 0\}$ is non-increasing.*
2. *Each of the Lévy processes of the decomposition (2.3.2) has a zero drift term, that is, $a_+ = a_- = 0$.*
3. *There exists $p > 1$, such that $\left|\int_{-\infty}^{0} (e^{xu} - 1)\,\mu_\alpha(dx)\right| \leq |u^{-p}|$ for all $\alpha \in \{+,-\}$ and $u \in \mathbb{C}$.*
4. *There exists $u > |\log(q_+) - \log(q_-)|$, such that $G_\alpha(x) < e^{-ux}$ for all $x > 0$ and $\alpha \in \{+,-\}$.*

An example of a process satisfying these assumptions is given in Section 4.3.

Proposition 4.2.1

Suppose that Assumption 4.2.1 holds. Then, for all $s \in \mathbb{C}^+$,

$$\mathbf{E}[A_\infty^s] = \lim_{n\to\infty} M_n(s)\mathbf{e},$$

where $\mathbf{e} = (1,1)^T$ and for all $n \in \mathbb{N}$,

$$M_n(s) := \left(\prod_{k=1}^{n} \frac{-F(k+s)}{k+s}\right) \operatorname{diag}\left(\left\{\left(-\left(\frac{F(n+1)}{n+1}\right)^{-1}\right)_{\alpha,\alpha}^{-s}\right\}_{\alpha\in E}\right)\left(\prod_{k=n}^{1}\left(\frac{-F(k)}{k}\right)^{-1}\right).$$

Before proving the proposition we will need some preliminary lemmas. For all $z \in \mathbb{C}$, set $H(z) := (z+1)\,(-F(z+1))^{-1}$ and notice that $\lim_{\Re(z)\to\infty} K_\alpha(z)H_{\alpha,\alpha}(z) = 1$ for each $\alpha \in E$.

The following lemma allows us to greatly simplify the calculations which show Assumption 4.1.1.8 holds for H.

Lemma 4.2.1

Suppose H satisfies Assumption 4.1.1 except Assumption 4.1.1.8 and that H^{-1} satisfies

Assumption 4.1.1.8, with constants $t_1, t_2 > 0$ such that $2t_2 + t_1 \in (0, 2)$. Then, H also satisfies Assumption 4.1.1.8.

Proof

By a standard manipulation of fractions, for all $z \in \mathcal{S}_{a,\infty}$ and $\alpha \in E$, we have

$$
\begin{aligned}
(H(z))_{\alpha,\alpha} &= \frac{(H(z))^{-1}_{-\alpha,-\alpha}}{(H(z))^{-1}_{++}(H(z))^{-1}_{--} - (H(z))^{-1}_{+-}(H(z))^{-1}_{-+}} \\
&= \frac{1}{(H(z))^{-1}_{\alpha,\alpha}} + \frac{1}{(H(z))^{-1}_{\alpha,\alpha}} \frac{(H(z))^{-1}_{+-}(H(z))^{-1}_{-+}}{\det(H(z)^{-1})} \\
&= \frac{1}{(H(z))^{-1}_{\alpha,\alpha}} \left(1 + \frac{H_{+,-}(z)H_{-,+}(z)}{\det(H(z))}\right).
\end{aligned}
$$

Then, by applying Assumption 4.1.1.1 and Assumption 4.1.1.2, there exists a constant $C_1 > 0$, such that

$$
(H(z))_{\alpha,\alpha} = \frac{1}{(H(z))^{-1}_{\alpha,\alpha}}(1 + C_1\omega(e^{-uz})).
$$

Hence, for $n \in \mathbb{N}$,

$$
\begin{aligned}
&\prod_{k=1}^{n}\left(1 + \frac{\Delta_s H_{\alpha,\alpha}(k)}{H_{\alpha,\alpha}(k+s)}\right) = \prod_{k=1}^{n}\left(\frac{H_{\alpha,\alpha}(k)}{H_{\alpha,\alpha}(k+s)}\right) \\
&= \left\{\prod_{k=1}^{n}\left(\frac{H^{-1}_{\alpha,\alpha}(k+s)}{H^{-1}_{\alpha,\alpha}(k)}\right)\right\}\left\{\prod_{k=1}^{n}\left(1 + C_1\omega\left(e^{-uk}\right)\right)\right\}\left\{\prod_{k=1}^{n}\left(1 + C_1\omega\left(e^{-u(k+s)}\right)\right)^{-1}\right\},
\end{aligned}
$$

where the last two products converge since $\sum_{k=1}^{n}\omega\left(e^{-uk}\right) < \infty$. Moreover, the last two products are bounded in absolute value, above and below, for $s \in \mathcal{S}_{a,b}$. That is, there exists $C_2, C_3 > 0$, such that for all $n \in \mathbb{N}$ and $s \in \mathcal{S}_{a,b}$,

$$
C_2 \le \left\{\prod_{k=1}^{n}\left(1 + C_1\omega\left(e^{-uk}\right)\right)\right\}\left\{\prod_{k=1}^{n}\left(1 + C_1\omega\left(e^{-u(k+s)}\right)\right)^{-1}\right\} \le C_3.
$$

Hence the convergence requirements of Assumption 4.1.1.8 of H follow from those for H^{-1}. To see the bounds of Assumption 4.1.1.8, notice that

$$
\frac{H^{-1}_{\alpha,\alpha}(k+s)}{H^{-1}_{\alpha,\alpha}(k)} = \left(1 + \frac{\Delta_s H^{-1}_{\alpha,\alpha}(k)}{H^{-1}_{\alpha,\alpha}(k+s)}\right)^{-1},
$$

hence, we have that

$$
\begin{aligned}
C_2\left|\left(H^{-1}_{\alpha,\alpha}(n)\right)^{-s}\prod_{k=1}^{n}\left(1 + \frac{\Delta_s H^{-1}_{\alpha,\alpha}(k)}{H^{-1}_{\alpha,\alpha}(k+s)}\right)^{-1}\right| &\le \left|H_{\alpha,\alpha}(n)^s\prod_{k=1}^{n}\left(1 + \frac{\Delta_s H_{\alpha,\alpha}(k)}{H_{\alpha,\alpha}(k+s)}\right)\right| \\
&\le C_3\left|\left(H^{-1}_{\alpha,\alpha}(n)\right)^{-s}\prod_{k=1}^{n}\left(1 + \frac{\Delta_s H^{-1}_{\alpha,\alpha}(k)}{H^{-1}_{\alpha,\alpha}(k+s)}\right)^{-1}\right|.
\end{aligned}
$$

Thus, Assumption 4.1.1.8 for H^{-1} is equivalent to Assumption 4.1.1.8 for H.

$\square$

Lemma 4.2.2

Assumptions 4.2.1, implies Assumption 4.1.1.8 holds for H^{-1}.

Proof

Suppose that $0 < a < b$. Then, observe that for each $\alpha \in E$ and $s \in \mathcal{S}_{a,b}$,

$$H^{-1}_{\alpha,\alpha}(N-1)^s \prod_{k=1}^{N-1} \left(1 + \frac{\Delta_s H^{-1}_{\alpha,\alpha}(k)}{H^{-1}_{\alpha,\alpha}(k+s)}\right) = K_\alpha(N)^s \prod_{k=2}^{N} \left(1 + \frac{\Delta_s K_\alpha(k)}{K_\alpha(k+s)}\right).$$

However, by a standard manipulation of fractions and Assumption 4.2.1.3, we have

$$\begin{aligned} 1 + \frac{\Delta_s K_\alpha(k)}{K_\alpha(k+s)} &= \frac{K_\alpha(k)}{K_\alpha(k+s)} = \left(\frac{k+s}{k}\right)\left(\frac{q_\alpha - \omega((k+s)^{-p})}{q_\alpha - \omega(k^{-p})}\right) \\ &= \left(\frac{k+s}{k}\right)\left(\frac{q_\alpha}{q_\alpha} + \frac{q_\alpha \omega(k^{-p})}{q_\alpha(q_\alpha - \omega(k^{-p}))} + \frac{\omega((k+s)^{-p})}{q_\alpha} + \frac{\omega(k^{-p}(k+s)^{-p})}{q_\alpha(q_\alpha - \omega(k^{-p}))}\right) \\ &= \left(1 + \frac{s}{k}\right)\left(1 + C_1 \omega(k^{-p})\right), \end{aligned}$$

for some constant $C_1 > 0$, for all $k \in \mathbb{N}$ and $s \in \mathcal{S}_{a,b}$. Using this within the product gives

$$H^{-1}_{\alpha,\alpha}(N-1)^s \prod_{k=1}^{N-1} \left(1 + \frac{\Delta_s H^{-1}_{\alpha,\alpha}(k)}{H^{-1}_{\alpha,\alpha}(k+s)}\right) = \left(K_\alpha(N)N\right)^s N^{-s} \left(\prod_{k=2}^{N}\left(1 + \frac{s}{k}\right)\right) \prod_{k=2}^{N} \left(1 + C_1\omega(k^{-p})\right).$$

Then, note that the rightmost product does not depend on s and, since $p > 1$, converges as $N \to \infty$. Thus, there is a constant $C_2 > 0$, such that, for all $N \in \mathbb{N}$,

$$C_2^{-1} \le \exp\left(-C_1 \sum_{k=1}^{\infty} k^{-p}\right) \le \prod_{k=2}^{N}\left(1 + C_1\omega(k^{-p})\right) \le \exp\left(+C_1 \sum_{k=1}^{\infty} k^{-p}\right) \le C_2.$$

Now consider $T > 0$. When taking the limit as $N \to \infty$, we have

$$\lim_{N\to\infty} K_\alpha(N)N = q_\alpha \qquad \text{and} \qquad \lim_{N\to\infty} N^{-s} \prod_{k=1}^{N}\left(1 + \frac{s}{k}\right) = \frac{1}{s\Gamma(s)}.$$

Then, since x^s is a continuous function of s, we have that $\left(K_\alpha(N)N\right)^s$ converges uniformly to q_α^s, as $N \to \infty$, within the compact domain $\{s \in \mathcal{S}_{a,b} : |\Im(s)| < T\}$. Moreover, it is known that the convergence to the Gamma function of the product is also uniform over $\{s \in \mathcal{S}_{a,b} : |\Im(s)| < T\} \subset \mathbb{C}^+$. Hence, the uniform convergence condition of Assumption 4.1.1.8 is met.

We now consider the bounds in Assumption 4.1.1.8. By Lemma A.1.2 followed by Lemma A.1.1, there exists $T, C_3, C_4, C_5 > 0$ and $N_1 \in \mathbb{N}$, such that for all $N > N_1$ and $s \in \mathcal{S}_{a,b}$ with $|\Im(s)| > T$,

$$C_3 \leq \left| N^{-s} \prod_{k=1}^{N} \left(1 + \frac{s}{n}\right) \right| \leq \frac{C_4}{|s\Gamma(s)|} \leq C_5 \exp\left(\frac{1}{2}\pi|\Im(s)|\right).$$

Pick $N_1 \in \mathbb{N}$, such that $\frac{1}{2}q_\alpha < K_\alpha(N)N < 2q_\alpha$ for all $N > N_1$. Then, for all $N > N_1$,

$$\begin{aligned}\frac{1}{2}C_2^{-1}C_3 \min(q_\alpha^a, q_\alpha^b) \leq \left| H_{\alpha,\alpha}^{-1}(N-1)^s \prod_{k=1}^{N-1} \left(1 + \frac{\Delta_s H_{\alpha,\alpha}^{-1}(k)}{H_{\alpha,\alpha}^{-1}(k+s)}\right) \right| \\ \leq 2C_2C_5 \max(q_\alpha^a, q_\alpha^b) \exp\left(\frac{1}{2}\pi|\Im(s)|\right).\end{aligned}$$

Hence, we have the bounds of Assumption 4.1.1.8. □

We are now able to prove Proposition 4.2.1.

Proof of Proposition 4.2.1

Provided Assumption 4.1.1 is satisfied for H and some $a, b \in \mathbb{R}$ such that $0 < a < b - 1$, we can apply Theorem 4.1.1 to obtain a matrix valued function $M : \mathcal{S}_{a,b} \to \mathbb{C}^{2\times 2}$ such that

$$M(s) = \lim_{n\to\infty} \left(\prod_{k=1}^{n} \frac{-F(k+s)}{k+s}\right) \begin{pmatrix} (H_{+,+}(n))^s & 0 \\ 0 & (H_{-,-}(n))^s \end{pmatrix} \left(\prod_{k=n}^{1} \left(\frac{-F(k)}{k}\right)^{-1}\right).$$

Then, by Proposition 4.1.1, we can extend M to all of $\mathbb{C}^+$. Moreover, by Theorem 4.1.1, M satisfies the assumptions of Theorem 4.1.2, hence $\mathbf{E}[A_\infty^s] = M(s)\mathbf{e}$, for all $s \in \mathbb{C}^+$.

We now check that there exists $a, b > 0$ such that Assumption 4.1.1 holds for H over $\mathcal{S}_{a,b}$. Assumption 4.1.1.1 is an immediate consequence of Assumption 4.2.1.4. From Assumption 4.2.1.3,

$$|K_\alpha(z)| = \left|\frac{q_\alpha - \omega(z^{-p})}{z}\right| \geq \frac{q_\alpha}{2|z|}, \tag{4.2.1}$$

for all $z \in \mathcal{S}_{a,\infty}$ provided $a > 0$ is sufficiently large. Combining this with Assumption 4.1.1.1 implies Assumption 4.1.1.2. Similarly,

$$\lim_{z\to\infty} \frac{H_{+,+}(z)}{H_{-,-}(z)} = \lim_{z\to\infty} \frac{K_{-,-}(z)}{K_{+,+}(z)} = \frac{q_-}{q_+} \in (0, \infty),$$

hence Assumption 4.1.1.3 holds with $v = |\log(q_-) - \log(q_+)|$. Assumption 4.1.1.4 is a consequence of the Lévy-Khintchine formula for subordinators. The matrix $H(z)$ is invertible whenever $z \neq -1$ and $\det(F(z+1)) \neq 0$, which is true for sufficiently large z by Assumption 4.1.1.2, hence Assumption 4.1.1.5 holds.

From the definition of H and a standard manipulation of fractions,

$$H_{\alpha,\alpha}(z) = \frac{-(z+1)}{\psi_\alpha(z+1) - q_\alpha} + \frac{(z+1)\omega(e^{-2u(z+1)})}{(\psi_+(z+1) - q_+)(\psi_-(z+1) - q_-) + \omega(e^{-2u(z+1)})},$$

where we notice that the second fraction converges to zero as $\Re(z) \to \infty$. Moreover, for $k \in \mathbb{N}$ and $s \in \mathcal{S}_{a,b}$,

$$\begin{aligned}
\left| \left(\frac{q_\alpha}{k+s} \right)^{-1} \left(\frac{q_\alpha - \psi_\alpha(k+s)}{k+s} \right) - 1 \right| &= \left| \frac{q_\alpha - \int_{-\infty}^0 \left(e^{(k+s)x} - 1\right) \mu_\alpha(dx)}{q_\alpha} - 1 \right| \\
&= \left| q_\alpha^{-1} \int_{-\infty}^0 \left(e^{(k+s)x} - 1\right) \mu_\alpha(dx) \right| \\
&\le q_\alpha^{-1} |k+s|^{-p} \le q_\alpha^{-1} |k|^{-p},
\end{aligned}$$

where the last line follows from Assumption 4.2.1.3 and the fact $s \in \mathcal{S}_{a,b}$. Then, since $p > 0$, we can conclude $H_{\alpha,\alpha}(k+s)\,(q_\alpha/(k+s)) \to 1$ as $k \to \infty$, uniformly for $s \in \mathcal{S}_{a,b}$. Hence, Assumption 4.1.1.6 holds.

The bounds of Assumption 4.1.1.7 follows from the uniform convergence of $(k+s)H_{\alpha,\alpha}(k+s)$ as $k \to \infty$ for all $s \in \mathcal{S}_{a,b}$, for a sufficiently large choice of $a > 0$. Assumption 4.1.1.9 follows from Assumption 4.2.1.4.

Notice that for $k \in \mathbb{N}$ and $s \in \mathcal{S}_{a,b}$,

$$\|H(k+s)^{-1}\| = \left\| \frac{1}{(k+s+1)} F(k+s+1) \right\|,$$

however, by Assumption 4.2.1.4, for each $\alpha \in E$,

$$\left| \frac{F_{\alpha,-\alpha}(k+s+1)}{k+s+1} \right| \le \frac{e^{-u(k+1+a)}}{|k+s+1|} \to 0 \qquad \text{as} \qquad |\Im(s)| \to \infty,$$

and by the Assumption 4.2.1

$$\left| \frac{F_{\alpha,\alpha}(k+s+1)}{k+s+1} \right| \le \left| \frac{q_\alpha}{k+s+1} \right| + |k+s+1|^{-p-1} \qquad \text{as} \qquad |\Im(s)| \to \infty.$$

Hence, by continuity of the entries of F, Assumption 4.1.1.10 holds. Then, Assumption 4.1.1.8 follows from Lemma 4.2.2 and Lemma 4.2.1. □

4.2.2 Spectrally Positive MAP

We now consider a second class of MAPs, which may have non-zero diffusion and drift coefficients, but are still spectrally one sided. We will continue to assume $E = \{+, -\}$ and make the following assumption.

Assumption 4.2.2

Suppose (J, ξ) is a MAP which satisfies the following conditions:

1. *There exists $u > 0$ such that $U_\alpha > u$, for all $\alpha \in E$.*
2. *Both Lévy processes of the decomposition (2.3.2) are spectrally positive, that is, $\mu_\pm(-\infty, 0) = 0$.*
3. *Both Lévy processes of the decomposition (2.3.2) have a non-zero diffusion component, that is, $\sigma_+^2 \neq 0 \neq \sigma_-^2$.*
4. *There exists $\delta > 0$ such that $|\log\left((1+\delta)\sigma_+^2\right) - \log\left((1-\delta)\sigma_-^2\right)| < u$.*
5. *The components of F are analytic.*
6. *$K_\alpha(-k) \neq 0$ for all $k \in \mathbb{N}$ and $\alpha \in E$.*
7. *There is a $p > -1$ such that $\left|\int_0^\infty (e^{xz} - 1)\, \mu_\alpha(dz)\right| \leq |x|^{-p}$, for all $x \in \mathbb{C}^-$ and $\alpha \in E$.*

Under these assumptions, the *Cramér type condition* is satisfied (see Section 2.3.2). Let $\lambda_2(z)$ denote the eigenvalue of $F(z)$ which is not $\lambda(z)$. Then, $\Re(\lambda_2(z)) < \lambda(z) < 0$ for all $z \in (0, \kappa)$. Hence, $F(z)$ is invertible for all $z \in (0, \kappa)$.

We also note that $F(z)$ exists for all $z \in \mathbb{C}^-$, because of Assumptions (2) and (4). Let $\theta = \min\{\Re(z) \,:\, z \in \mathbb{C}, \det(F(z)) = 0\}$. We can choose $a, b \in \mathbb{R}^+$, such that $a < b - 1$ and $-a < \theta$. Then, $F(z)$ is invertible for all $z \in \mathcal{S}_{-b,-a}$. Moreover, $\lim_{z\to\infty} -z^{-1}F_{\alpha,\alpha}(-z) = -\infty$ for each $\alpha \in E$, so we can choose $a > 1$ such that $-z^{-1}F_{\alpha.\alpha}(-z)$ is non-zero for all $\alpha \in E$ and $z \in \mathbb{C}$ with $\Re(z) > a$.

Proposition 4.2.2

Suppose that Assumption 4.2.2 holds. Then, for all $s \in \mathcal{S}_{-\infty,\kappa}$,

$$\mathbf{E}[A_\infty^s] = \mathcal{M}(s)\mathbf{m},$$

where $\mathbf{m} := \mathbf{E}[A_\infty^{-1}] = F'(0)\mathbf{e} + F(0)\mathbf{E}[\log A_\infty]$ and

$$\mathcal{M}(s) := \lim_{n\to\infty} \left(\prod_{k=0}^{n} \left(\frac{F(-k+s)}{k-s}\right)^{-1}\right) \operatorname{diag}\left(\left\{K_\alpha(-n-1)^{-s-1}\right\}_{\alpha\in E}\right) \left(\prod_{k=n+1}^{1} \frac{F(-k)}{k}\right),$$

for all $s \in \mathcal{S}_{-\infty,\kappa} \setminus \mathbb{Z}$ and is the analytic continuation for all $s \in \mathcal{S}_{-\infty,\kappa} \cap \mathbb{Z}$.

For consistency with Theorem 4.1.1, let $M(s) := \mathcal{M}(-s-1)$ and define a matrix valued function $H : \mathcal{S}_{a,b} \to \mathbb{C}^{2\times 2}$ by

$$H(s) = (s+1)^{-1}F(-s-1). \tag{4.2.2}$$

For $R \in \mathbb{N}$, $\alpha \in E$ and $s \in \mathcal{S}_{a,b}$, also define the function

$$P_\alpha(s,R) := \prod_{n=1}^{R} \left(1 - \frac{\Delta_s K_\alpha(-n)}{K_\alpha(-n)}\right) = \prod_{n=1}^{R} \frac{K_\alpha(-n+s)}{K_\alpha(-n)}.$$

We first prove some preliminary lemmas.

Lemma 4.2.3

There exist constants $C_1, C_2 > 0$, such that for each $s \in \mathbb{C}^-$, $n \in \mathbb{N}$ and $\alpha \in \{+,-\}$,

$$|\Delta_s K_\alpha(-n)| \le \frac{\sigma_\alpha^2}{2}|s| + C_1 \frac{1}{n^r} \qquad \text{and} \qquad \frac{\Delta_s K_\alpha(-n)}{K_\alpha(-n)} = \frac{s}{n} + C_2 \omega\left(\frac{1+|s|}{n^{1+r}}\right),$$

where $r := \min(1, 1+p)$.

Proof

Suppose $\alpha \in E$ and $s \in \mathbb{C}^-$, then from the Lévy-Khintchine formula,

$$\Delta_s K_\alpha(n) = \frac{q_\alpha s}{n(n+s)} + \frac{\sigma_\alpha^2}{2}s + \int_0^\infty \left(\frac{(e^{(n+s)z}-1)}{n+s} - \frac{(e^{nz}-1)}{n}\right)\mu_\alpha(dz).$$

Now suppose $s \in \mathbb{C}^-$, then for $n \in \mathbb{N}_0$, $|n-s| \ge |s|$ and $|n-s| \ge n$, so

$$\begin{aligned} |\Delta_s K_\alpha(-n)| &\le \frac{q_\alpha |s|}{|n(n-s)|} + \frac{\sigma_\alpha^2}{2}|s| + \left|\int_0^\infty \left(\frac{e^{(-n+s)z}-1}{s-n}\right)\mu_\alpha(dz)\right| + \left|\int_0^\infty \left(\frac{e^{-nz}-1}{n}\right)\mu_\alpha(dz)\right| \\ &\le \frac{q_\alpha}{n} + \frac{\sigma_\alpha^2}{2}|s| + |s-n|^{-p-1} + n^{-p-1}, \end{aligned}$$

hence, the first claims holds.

Now consider the second claim. From the assumptions, there also exists $C_2 > 0$, such that for $n \in \mathbb{N}$ and $\alpha \in E$,

$$K_\alpha(-n) = \frac{\sigma_\alpha^2}{2}n - a_\alpha - \frac{1}{n}\left(q_\alpha + \int_0^\infty (1-e^{-nz})\mu_\alpha(dz)\right) = \frac{\sigma_\alpha^2}{2}n + \omega(C_2). \qquad (4.2.3)$$

Then, by a standard manipulation of fractions, there exits $C_3 > 0$ such that

$$\frac{1}{K_\alpha(-n)} = \frac{1}{\frac{\sigma_\alpha^2}{2}n + \omega(C_2)} = \frac{2}{\sigma_\alpha^2 n} + \frac{\omega(C_2)}{\frac{\sigma_\alpha^2}{2}n\left(\frac{\sigma_\alpha^2}{2}n + \omega(C_2)\right)} = \frac{2}{\sigma_\alpha^2 n} + C_3\omega\left(\frac{1}{n^2}\right).$$

Combining these two results gives the second claim:

$$\frac{\Delta_s K_\alpha(-n)}{K_\alpha(-n)} = \left(s\frac{\sigma_\alpha^2}{2} + C_1\omega\left(\frac{1}{n^r}\right)\right)\left(\frac{2}{\sigma_\alpha^2}\frac{1}{n} + C_3\omega\left(\frac{1}{n^2}\right)\right) = \frac{s}{n} + C_4\omega\left(\frac{1+|s|}{n^{1+r}}\right),$$

where $r := \min(p+1, 1)$ and $C_4 > 0$ is some constant. $\square$

The following is a technical lemma providing a bound which will be useful in the later calculations.

Lemma 4.2.4

For each $R \in \mathbb{N}$, there exist constants $T, C_1, C_2 > 0$, such that

$$C_1 |\Im(s)|^R \leq P_\alpha(s, R) \leq C_2 |\Im(s)|^R,$$

for all $\alpha \in E$ and $-s \in \mathcal{S}_{a,b}$ with $|\Im(s)| > T$.

Proof

Fix $R \in \mathbb{N}$. From (4.2.3), there exists $C_3 > 0$, such that $K_\alpha(z) = -\frac{\sigma_\alpha^2}{2} z + C_3 \omega(1)$, for each $z \in \mathbb{C}^-$. Hence, we have the asymptotic equivalence $K_\alpha(n+s) \sim \frac{\sigma_\alpha^2}{2} s$ as $|\Im(s)| \to \infty$. Hence, for all $\epsilon > 0$, if $T > 0$ is sufficiently large and $-s \in \mathcal{S}_{a,b}$ with $|\Im(s)| > T$, then

$$(1-\epsilon)\frac{\sigma_\alpha^2}{2}|\Im(s)| \leq |K_\alpha(n+s)| \leq (1+\epsilon)\frac{\sigma_\alpha^2}{2}|\Im(s)|,$$

hence,

$$C_1 |\Im(s)|^R := \frac{(1-\epsilon)^R |\Im(s)|^R \left(\frac{\sigma_\alpha^2}{2}\right)^R}{\prod_{n=1}^R K_\alpha(-n)} \leq P(s,R) \leq \frac{(1+\epsilon)^R |\Im(s)|^R \left(\frac{\sigma_\alpha^2}{2}\right)^R}{\prod_{n=1}^R K_\alpha(-n)} =: C_2 |\Im(s)|^R,$$

where the constants $C_1, C_2 > 0$ are independent of s but depend upon R. $\square$

Lemma 4.2.5

For any $T > 0$, the limit

$$\lim_{N\to\infty} K_\alpha(-N)^{-z} \prod_{n=1}^{N} \left(1 + \frac{\Delta_z K_\alpha(-n)}{K_\alpha(z-n)}\right), \tag{4.2.4}$$

exists for all $z \in \mathcal{S}_{-b,-a}$ and the convergence is uniform on $\{z \in \mathcal{S}_{-b,-a} : |\Im(s)| < T\} \subset \mathbb{C}^-$.

Proof

Fix $T > 0$. We first consider the reciprocal of (4.2.4), which we write in the form

$$K_\alpha(-N)^z \prod_{n=1}^{N} \left(1 + \frac{\Delta_z K_\alpha(-n)}{K_\alpha(z-n)}\right)^{-1} = K_\alpha(-N)^z \exp\left(\sum_{n=1}^{N} \log\left(1 - \frac{\Delta_z K_\alpha(-n)}{K_\alpha(-n)}\right)\right).$$

Then, notice that there exists an $N_1 \in \mathbb{N}$, such that for all $n > N_1$ and $z \in \mathcal{S}_{-b,-a}$ with $|\Im(s)| < T$, we have

$$C_2 \left|\frac{1+|z|}{n^{1+r}}\right| < \frac{1}{2}\left|1 - \frac{z}{n}\right|,$$

where $C_2 > 0$ is the constant from Lemma 4.2.3. Taking the Taylor expansion of the result of Lemma 4.2.3 about $1 - \frac{z}{n}$, we obtain

$$\begin{aligned}\log\left(1 - \frac{\Delta_z K_\alpha(-n)}{K_\alpha(-n)}\right) &= \log\left(1 - \frac{z}{n}\right) + C_2 \omega\left(\frac{1+|z|}{n^{1+r}}\right)\omega\left(\sup_{|x|<C_2(1+|z|)n^{-(1+r)}} \frac{1}{1 - \frac{z}{n} + x}\right) \\ &= \log\left(1 - \frac{z}{n}\right) + 2C_2 \omega\left(\frac{1+|z|}{n^{1+r}}\right).\end{aligned}$$

Thus, we have

$$
\begin{aligned}
&K_\alpha(-N)^z \prod_{n=1}^{N} \left(1 + \frac{\Delta_z K_\alpha(-n)}{K_\alpha(z-n)}\right)^{-1} \\
&\quad = K_\alpha(-N)^z \prod_{n=1}^{N_1-1} \left(1 - \frac{\Delta_z K_\alpha(-n)}{K_\alpha(-n)}\right) \exp\left(\sum_{n=N_1}^{N} \left(\log\left(1 - \frac{z}{n}\right) + 2C_2\omega\left(\frac{1+|z|}{n^{1+r}}\right)\right)\right) \\
&\quad = \left(\frac{K_\alpha(-N)}{N}\right)^z \prod_{n=1}^{N_1-1} \left\{\left(1 - \frac{\Delta_z K_\alpha(-n)}{K_\alpha(-n)}\right)\left(1 - \frac{z}{n}\right)^{-1}\right\} \\
&\qquad \times \exp\left(\sum_{n=N_1}^{N} 2C_4\omega\left(\frac{1+|z|}{n^{1+r}}\right)\right)\left\{N^z \prod_{n=1}^{N}\left(1 - \frac{z}{n}\right)\right\}.
\end{aligned}
$$

The first term converges uniformly on $\{s \in \mathcal{S}_{a,b} : |\Im(s)| < T\}$, since $\lim_{N\to\infty} K_\alpha(-N)/N = \frac{\sigma_\alpha^2}{2}$ and x^z is an analytic function on any domain excluding 0. The middle term does not depend on N and is non-zero. Moreover, we know that

$$
\lim_{N\to\infty} N^z \prod_{n=1}^{N}\left(1 - \frac{z}{n}\right) = \frac{1}{-z\Gamma(-z)},
$$

which is bounded and converges uniformly in $\{z \in \mathcal{S}_{-b,-a} : |\Im(z)| < T\}$. Then, since the limit of each of these terms is non-zero, the convergence of (4.2.4) must also be uniform in $\{z \in \mathcal{S}_{-b,-a} : |\Im(z)| < T\}$. □

The next lemma will be useful when proving the bounds of Assumption 4.1.1.8.

Lemma 4.2.6

For all $t_1, t_2 > 0$, there exist constants $C_1, C_2, T, R > 0$, such that, for all $N > R$, $s \in \mathcal{S}_{-b,-a} \subset \mathbb{C}^-$ with $|\Im(s)| > T$ and all $\alpha \in \{+,-\}$,

$$
C_1 \exp\left(-\left(t_1 + \frac{1}{2}\right)\pi|\Im(s)|\right) \le \left|K_\alpha(-N)^{-s} \prod_{n=1}^{N}\left(1 + \frac{\Delta_s K_\alpha(-n)}{K_\alpha(s-n)}\right)\right| \le C_2 \exp\left(t_2\pi|\Im(s)|\right).
$$

Moreover, there exists a constant $C_3 > 0$ such that

$$
\lim_{N\to\infty} K_\alpha(-N)^{-s} \prod_{n=1}^{N}\left(1 + \frac{\Delta_s K_\alpha(-n)}{K_\alpha(s-n)}\right) \le C_3 \exp\left(\left(t_2 - \frac{\pi}{2}\right)|\Im(s)|\right),
$$

for all $s \in \mathcal{S}_{-b,-a}$ with $|\Im(s)| > T$.

Proof

Suppose $s \in \mathcal{S}_{-b,-a}$. Then, consider the representation

$$
K_\alpha(-N)^s \prod_{n=1}^{N}\left(1 + \frac{\Delta_s K_\alpha(-n)}{K_\alpha(s-n)}\right)^{-1} = K_\alpha(-N)^s \exp\left(\sum_{n=1}^{N} \log\left(1 - \frac{\Delta_s K_\alpha(-n)}{K_\alpha(-n)}\right)\right).
$$

From Lemma 4.2.3, we have that, for some $C_4 > 0$,

$$\frac{\Delta_s K_\alpha(-n)}{K_\alpha(-n)} = \frac{s}{n} + C_4 \omega\left(\frac{1+|s|}{n^{1+r}}\right).$$

Fix $T > 1+b$. Then, since $s \in \mathcal{S}_{-b,-a} \subset \mathbb{C}^-$, we have $\left|1 - \frac{s}{n}\right| \geq \frac{|s|}{2n}$, whilst for all $s \in \mathcal{S}_{-b,-a}$ with $|\Im(s)| > T$,

$$1 + |s| \leq 1 + |\Re(s)| + |\Im(s)| \leq 1 + b + |\Im(s)| \leq 2|\Im(s)| \leq 2|s|.$$

Thus, for all $s \in \mathcal{S}_{-b,-a}$ with $|\Im(s)| > T$ and $n \in \mathbb{N}$,

$$\left|\frac{1 - \frac{s}{n}}{C_4(1+|s|)n^{-(1+r)}}\right| \geq \frac{n^r}{4C_4}.$$

Then, for all $n > (8C_4)^{1/r} =: R$ and all $x \in \mathbb{C}$, such that $|x| < C_4(1+|s|)n^{-(1+r)}$, we have $\left|\frac{1-\frac{s}{n}}{x}\right| \geq \frac{n^r}{4C_4} \geq 2$, which can be rearranged to show $|x| \leq \frac{1}{2}\left|1 - \frac{s}{n}\right|$. Using the reverse triangle inequality, we deduce that for $s \in \mathbb{C}^-$

$$\left|1 - \frac{s}{n} + x\right| \geq \frac{1}{2}\left|1 - \frac{s}{n}\right| \geq \frac{1}{2}.$$

Then, by considering the Taylor expansion of $\log(\cdot)$ about $1 - \frac{s}{n}$, we have for $n > R$ and $s \in \mathcal{S}_{-b,-a}$ such that $|\Im(s)| > T$,

$$\begin{aligned}
\log\left(1 - \frac{\Delta_s K_\alpha(-n)}{K_\alpha(-n)}\right) &= \log\left(1 - \frac{s}{n}\right) + C_4\omega\left(\frac{1+|s|}{n^{1+r}}\right)\omega\left(\sup_{|x|<C_4(1+|s|)n^{-2}} \frac{1}{1 - \frac{s}{n} + x}\right) \\
&= \log\left(1 - \frac{s}{n}\right) + 2C_4\omega\left(\frac{1+|s|}{n^{1+r}}\right).
\end{aligned}$$

Supposing $|\Im(s)| > b/\sqrt{3}$ ensures that $|s| < 2|\Im(s)|$ and so we set $T := \max\left(\frac{b}{\sqrt{3}}, |b|+1\right)$. Then for all $N > R$ and $s \in \mathcal{S}_{-b,-a}$ with $|\Im(s)| > T$, we obtain

$$\begin{aligned}
&K_\alpha(-N)^s \prod_{n=1}^{N}\left(1 + \frac{\Delta_s K_\alpha(-n)}{K_\alpha(s-n)}\right)^{-1} \\
&\quad = K_\alpha(-N)^s P_\alpha(s,R)\exp\left(\sum_{n=R+1}^{N}\log\left(1 - \frac{\Delta_s K_\alpha(-n)}{K_\alpha(-n)}\right)\right) \\
&\quad = \frac{K_\alpha(-N)^s}{N^s} P_\alpha(s,R)\left(N^s \prod_{n=R+1}^{N}\left(1 - \frac{s}{n}\right)\right)\exp\left(C_6 \sum_{n=R+1}^{N}\omega\left(\frac{1+|s|}{n^{1+r}}\right)\right),
\end{aligned}$$

where $C_6 := 2C_4$. However, $|s| \leq |\Re(s)| + |\Im(s)| \leq |b| + |\Im(s)|$, hence

$$\left|\sum_{n=R+1}^{N}\omega\left(\frac{1+|s|}{n^{1+r}}\right)\right| \leq (1 + |b| + |\Im(s)|)\sum_{n=R+1}^{\infty} n^{-(1+r)}.$$

Then, for any $t > 0$, by taking R sufficiently large, $\sum_{n=R+1}^{\infty} n^{-(1+r)} < tC_6^{-1}$, thus

$$\left| \sum_{n=m \vee R+1}^{N} \omega\left(\frac{1+|s|}{n^{1+r}}\right) \right| = tC_6^{-1}(1+|b|+|\Im(s)|).$$

Substituting this into the product gives

$$\begin{aligned} &K_\alpha(-N)^s \prod_{n=1}^{N} \left(1 + \frac{\Delta_s K_\alpha(-n)}{K_\alpha(s-n)}\right)^{-1} \\ &= \frac{K_\alpha(-N)^s}{N^s} P(s,R) \left(N^s \prod_{n=1}^{N} \left(1 - \frac{s}{n}\right)\right) \left(\prod_{n=1}^{R} \left(1 - \frac{s}{n}\right)^{-1}\right) \exp\left(\omega(t(1+|b|+|\Im(s)|))\right). \end{aligned} \tag{4.2.5}$$

Taking absolute values, using Lemma A.1.2 and since $K_\alpha(-N)/N$ converges as $N \to \infty$ gives, for all $N \in \mathbb{N}$ such that $N > R$,

$$\left| K_\alpha(-N)^s \prod_{n=1}^{N} \left(1 + \frac{\Delta_s K_\alpha(-n)}{K_\alpha(s-n)}\right)^{-1} \right| \leq \frac{C_7 e^{t|\Im(s)|}}{|s\Gamma(-s)|} |P_\alpha(s,R)| \left| \prod_{n=1}^{R} \left(1 - \frac{s}{n}\right)^{-1} \right|,$$

for some constant $C_7 > 0$. Then, for all $s \in \mathcal{S}_{a,b}$ with $|\Im(s)| > T$ and $N \in \mathbb{N}$ such that $N > R$, using Lemma 4.2.4 and the fact $\left|1 - \frac{s}{n}\right| \geq 1$, we have

$$\left| K_\alpha(-N)^s \prod_{n=1}^{N} \left(1 + \frac{\Delta_s K_\alpha(-n)}{K_\alpha(s-n)}\right)^{-1} \right| \leq \frac{C_8 |\Im(s)|^R e^{t|\Im(s)|}}{|s\Gamma(-s)|} \leq \frac{C_9 e^{t|\Im(s)|}}{|s\Gamma(-s)|},$$

for some constants $C_8, C_9 > 0$.

Now consider the reciprocal of (4.2.5). For $N \in \mathbb{N}$ such that $N > R$,

$$\begin{aligned} &K_\alpha(-N)^{-s} \prod_{n=1}^{N} \left(1 + \frac{\Delta_s K_\alpha(-n)}{K_\alpha(s-n)}\right) = \\ &\frac{N^s}{K_\alpha(-N)^s} P_\alpha(s,R)^{-1} \left(N^{-s} \prod_{n=1}^{N} \left(1 - \frac{s}{n}\right)^{-1}\right) \prod_{n=1}^{R} \left(1 - \frac{s}{n}\right) \exp\left(\omega(t(1+|b|+|\Im(s)|))\right). \end{aligned} \tag{4.2.6}$$

Then, taking absolute values, using Lemma A.1.2 and uniform convergence of $N/K_\alpha(-N)$, there exists $C_{10} > 0$ such that, for sufficiently large N and all $s \in \mathcal{S}_{-b,-a}^T$,

$$\left| K_\alpha(-N)^{-s} \prod_{n=1}^{N} \left(1 + \frac{\Delta_s K_\alpha(-n)}{K_\alpha(s-n)}\right) \right| \leq C_{10} e^{\omega(t|\Im(s)|)} \left| P(s,R)^{-1} \prod_{n=1}^{R} \left(1 - \frac{s}{n}\right) \right|.$$

Then, using Lemma 4.2.4 and the fact $\left|1 - \frac{s}{n}\right| \leq 2|\Im(s)|$ for $s \in \mathcal{S}_{-b,-a}^T$ with $T > 1+b$, we have, for some constant $C_{11} > 0$,

$$\left| K_\alpha(-N)^{-s} \prod_{n=1}^{N} \left(1 + \frac{\Delta_s K_\alpha(-n)}{K_\alpha(s-n)}\right) \right| \leq C_{11} \exp(t|\Im(s)|)|\Im(s)|^{R-R} = C_{11} \exp\left(t|\Im(s)|\right).$$

Moreover, if we consider the limit as $N \to \infty$ of (4.2.6), then there exist constants $C_{12}, C_{13} > 0$ such that for all $s \in \mathcal{S}_{-b,-a}$ with $|\Im(s)| > T$,

$$\lim_{N\to\infty} K_\alpha(-N)^{-s} \prod_{n=1}^{N} \left(1 + \frac{\Delta_s K_\alpha(-n)}{K_\alpha(s-n)}\right) \leq C_{12} |\Im(s)|^{-R} |\Gamma(-s)| \exp(\alpha|\Im(s))$$
$$\leq C_{13} \exp\left(\left(t - \frac{\pi}{2}\right) |\Im(s)|\right),$$

where the second inequality follows from Lemma A.1.1. □

Lemma 4.2.7

Under the Assumption 4.2.2, Assumption 4.1.1.8 holds for H.

Proof

Suppose $s \in \mathcal{S}_{a,b} \subset \mathbb{C}^+$ and let $z = -s \in \mathcal{S}_{-b,-a} \subset \mathbb{C}^-$. Notice that $H_{\alpha,\alpha}(s) = K_\alpha(z-1)$ and $\Delta_s H_{\alpha,\alpha}(k) = \Delta_z K_\alpha(-k-1)$ for all $k \in \mathbb{N}$ and $\alpha \in E$. Then,

$$H_{\alpha,\alpha}(N)^s \prod_{k=1}^{N} \left(1 + \frac{\Delta_s H_{\alpha,\alpha}(k)}{H_{\alpha,\alpha}(k+s)}\right) = K_\alpha(-N-1)^{-z} \prod_{k=1}^{N} \left(1 + \frac{\Delta_z K_\alpha(-k-1)}{K_\alpha(z-k-1)}\right).$$

Hence, by Lemma 4.2.6, for any $t_1, t_2 > 0$ and $\alpha \in E$ there exists $C_1, C_2, R, T > 0$, such that if $s \in \mathcal{S}_{a,b}$ with $|\Im(s)| > T$ and $N \in \mathbb{N}$ with $N > R$, then

$$C_1 \exp\left(-\left(t_1 + \frac{1}{2}\right)\pi|\Im(s)|\right) \leq \left| H_{\alpha,\alpha}(N)^s \prod_{k=1}^{N} \left(1 + \frac{\Delta_s H_{\alpha,\alpha}(k)}{H_{\alpha,\alpha}(k+s)}\right)\right| \leq C_2 \exp\left(t_2 \pi |\Im(s)|\right).$$

Moreover, by Lemma 4.2.5, this converges uniformly on $\{s \in \mathcal{S}_{a,b} : |\Im(s)| < T\}$ to something non-zero. Hence, the expression is bounded on the compact domain $\{s \in \mathcal{S}_{a,b} : |\Im(s)| < T\}$, so we can choose the constants $C_1, C_2 > 0$ such that the inequality holds on all of $\mathcal{S}_{a,b}$ for sufficiently large N. Thus, Assumption 4.1.1.8 holds. □

We are now able to prove Proposition 4.2.2.

Proof of Proposition 4.2.2

We first show that $\mathbf{E}[A_\infty^s] = \mathcal{M}(s)\mathbf{m}$ in the strip $\mathcal{S}_{-b,-a}$. This strip was chosen such that $-b+1 < -a < 0$ and $\det(F(s)) \neq 0$, for all $s \in \mathcal{S}_{-\infty,-a}$. Recall the function $H : \mathbb{C} \to \mathbb{C}^{2\times 2}$ from (4.2.2). Then, $M : \mathcal{S}_{a,b} \to \mathbb{C}^{2\times 2}$ can be written in terms of H as

$$M(s) = \mathcal{M}(-s-1) = \lim_{n\to\infty} \left(\prod_{k=0}^{n} H(k+s)^{-1}\right) \begin{pmatrix} H_{++}(n)^s & 0 \\ 0 & H_{--}(n)^s \end{pmatrix} \left(\prod_{k=n}^{0} H(k)\right).$$

Provided Assumption 4.1.1 is satisfied for H in the strip $\mathcal{S}_{a,b}$, by Theorem 4.1.1, the matrix

$M(z)$ exists for all $z \in \mathcal{S}_{a,b} \subset \mathbb{C}^+$ and has the properties:

$$\begin{aligned}
M(z+1) &= H(z)M(z) = (z+1)^{-1}F(-z-1)M(z); \\
M(0) &= I; \\
\|M(z)^{-1}\| &\leq C\exp(t\pi|\Im(z)|); \\
\|M(z)\| &\leq C\exp(t\pi|\Im(z)|);
\end{aligned}$$

for some $C > 0$ and $t \coloneqq t_1 + t_2$, where t_1, t_2 are the constants from Assumption 4.1.1.8. Moreover, by Proposition 4.1.1, M can be extended to $\mathcal{S}_{-\kappa-1,\infty}$. Hence, $\mathcal{M}(s) = M(-s-1)$ satisfies the conditions of Theorem 4.1.2 in the strip $\mathcal{S}_{-b+1,-a+1} \subset \mathbb{C}^-$ and so $\mathbf{E}[A_\infty^s] = \mathcal{M}(s)\mathbf{m}$, for all $s \in \mathcal{S}_{-\infty,\kappa}$.

We will therefore show that that the matrix valued function H satisfies the Assumption 4.1.1 on $\mathcal{S}_{a,b} \subset \mathbb{C}^+$, for a suitable choice of $a, b > 0$.

Assumption 4.1.1.1 follows immediately from Assumption 4.2.2.1.

The constant a was chosen such that $H_{\alpha,\alpha}(z) > 0$ for all $z \in \mathcal{S}_{a,\infty}$. Then, since $|H_{\alpha,\alpha}(z)| \to \infty$ as $|\Im(z)| \to \infty$, there exists a lower bound on $|H_{\alpha,\alpha}(z)|$ for all $z \in \mathcal{S}_{a,\infty}$. Hence, Assumption 4.1.1.2 follows from Assumption 4.2.2.1.

Since $\lim_{z\to\infty} \frac{H_{+,+}(z)}{H_{-,-}(z)} = \frac{\sigma_+^2}{\sigma_-^2} \in (0,\infty)$, Assumption 4.1.1.3 holds. Assumption 4.1.1.4 follows from the Lévy-Khintchine results. By the restriction that $-a < \theta$, Assumption 4.1.1.5 holds.

To see Assumption 4.1.1.6, first notice that

$$\begin{aligned}
\left|\frac{H_{\alpha,\alpha}(k-1)}{k\frac{\sigma_\alpha^2}{2}} - 1\right| &= \left|\frac{q_\alpha - a_\alpha k - \int_0^\infty \left(e^{-kz}-1\right)\mu_\alpha(dz)}{k^2\frac{\sigma_\alpha^2}{2}}\right| \\
&\leq \frac{q_\alpha}{k^2\frac{\sigma_\alpha^2}{2}} + \frac{|a_\alpha|}{k\frac{\sigma_\alpha^2}{2}} + \frac{\left|\int_0^\infty (e^{-kz}-1)\mu_\alpha(dz)\right|}{k^2\frac{\sigma_\alpha^2}{2}} \leq C(k^{-1} + k^{-2-p}),
\end{aligned}$$

hence, $\left(k\frac{\sigma^2}{2}\right)^{-1} H_{\alpha,\alpha}(k) \to 1$ as $k \to \infty$. Moreover, for any $s \in \mathcal{S}_{a,b}$, we have

$$\frac{H_{\alpha,\alpha}(k+s-1)}{(k+s)\frac{\sigma_\alpha^2}{2}} \to 1 \qquad \text{as} \qquad k \to \infty,$$

uniformly for $s \in \mathcal{S}_{a,b}$. Combining the limits for $H_{\alpha,\alpha}(k+s)$ and $H_{\alpha,\alpha}(k)$ then gives Assumption 4.1.1.6.

Assumption 4.1.1.7 follows from the Lévy-Khintchine formula and our choice of a such that $|H_{\alpha,\alpha}(z)|$ is bounded below for all $\Re(z) > a$. Assumption 4.1.1.8 is given by Lemma 4.2.7. Assumption 4.1.1.9 follows from Assumption 4.2.2.4.

As $\Im(s) \to \pm\infty$, $H_{\alpha,\alpha}(s) \sim \frac{\sigma_\alpha^2}{2} s$, whilst $H_{\alpha,-\alpha}(s) \to 0$ and so is bounded. Thus, $\|H(s)^{-1}\| \to 0$ as $\Im(s) \to \pm\infty$. Then, by continuity it is bounded over $\mathcal{S}_{a,b}$, hence Assumption 4.1.1.10 holds. $\square$

4.3 Example

We are interested in finding examples where the results of this chapter can be used to compute the Mellin transform of the density of A_∞ explicitly. The easiest way to do this is to find cases where $\mathbf{m}$ is an eigenvector of $F(-x)/x$, for all $x \in \mathbb{C}$.

Example 4.3.1 (Split Exponential Jumps)
Consider the setting of Section 4.2.1. Then $\mathbf{m} = (1,1)^T$ is a right eigenvector of $F(x)$ for all $x \in \mathbb{C}^+$ if and only if

$$\psi_+(x) + q_+ \left(G_+(x) - 1\right) = \psi_-(x) + q_- \left(G_-(x) - 1\right). \tag{4.3.1}$$

In this case, $\mathbf{m}$ corresponds to the eigenvalue $\left(\psi_+(x) + q_+ G_+(x) - q_+\right)/x$ of $F(x)/x$. Using this within Proposition 4.2.1, for all $s \in \mathbb{C}^+$, we have

$$\begin{aligned}
&\mathbf{E}[A_\infty^s] = M(s)\mathbf{m} \\
&= \lim_{n\to\infty} \left(\frac{q_+ - \psi_+(n) - q_+ G_+(n)}{n}\right)^{-s} \prod_{k=1}^{n} \left(\frac{k}{k+s}\right) \left(\frac{\psi_+(k+s) + q_+ G_+(k+s) - q_+}{\psi_+(k) + q_+ G_+(k) - q_+}\right) \mathbf{m},
\end{aligned}$$

which corresponds to the generalised Gamma functions studied in [44] and [53]. In particular, from [44, Theorem 2.11] the tail asymptotics can be obtained.

To obtain a more explicit result, further assume that the two Lévy processes are compound Poisson processes (so $a_\pm = \sigma_\pm = 0$). Also, suppose that $q_+ = q_- =: q$ and the Lévy measure of $\xi^{(\pm)}$ is given by $\mu_\pm := q\nu_\mp$. Then, $\psi_\pm(x) = q(G_\mp(x) - 1)$. Moreover, suppose that $\mu_+(x) + \mu_-(x) = 2e^x$, for all $x \in (-\infty, 0)$. Then, $\frac{1}{2}(\mu_+ + \mu_-)(-x)$ is the density of an exponentially distributed random variable with rate 1. For all $x > 0$, a direct calculation shows

$$\psi_\pm(x) + qG_\pm(x) - q_\pm = q\left(G_\mp(x) + G_\pm - 2\right) = q \int_{-\infty}^{0} e^{xz}(\nu_\pm(dz) + \nu_\mp(dz)) - 2q = -\frac{2qx}{x+1}.$$

Then, it follows that

$$\begin{aligned}
\mathbf{E}[A_\infty^s] &= (2q)^{-s} \lim_{n\to\infty} n^s \left\{\prod_{k=1}^{n} \left(\frac{k}{k+s}\right) \left(\frac{2q(k+s)}{2qk}\right) \left(\frac{k+1}{k+s+1}\right)\right\} \mathbf{m} \\
&= (2q)^{-s} \frac{1+s}{1} \lim_{n\to\infty} (n+1)^s \prod_{k=1}^{n+1} \left(\frac{k}{k+s}\right) \mathbf{m}.
\end{aligned}$$

Hence, by identifying the Gamma function,

$$\mathbf{E}[A_\infty^s] = (2q)^{-s}(1+s)s\Gamma(s)m = (2q)^{-s}\Gamma(s+2)\mathbf{m}.$$

Notice that in this example $\mathbb{E}_\alpha[A_\infty^s]$ is the same, regardless of the value of $\alpha \in E$. From this it is possible to recover the density of A_∞, via a Mellin inversion. Let p denote the density of A_∞ and $\mathcal{M}$ the Mellin transform operator. Then, $\mathcal{M}\{p\}(s) = (2q)^2(2q)^{-(s+1)}\Gamma(s+1)$, thus

$$p(x) = \mathcal{M}^{-1}\{4q^2(2q)^{-(s+1)}\Gamma(s+1)\}(x) = 4q^2x\mathcal{M}^{-1}\{(2q)^{-s}\Gamma(s)\}(x) = 4q^2xe^{-2qx}.$$

Hence, A_∞ has a *Gamma distribution* with *shape* parameter 2 and *rate* parameter $2q$.

Chapter 5

Tails of the Exponential Functional

The right tails of a random variable determine which positive moments exist and whether it is a member of the classes of *heavy tailed*, *long tailed* or *subexponential* random variables. For each of these classes of random variable there is a large body of literature, for example see [24]

In the case of a strictly decreasing MAP with non-zero drift, A_∞ is bounded and so the right tail is zero. However, if the MAP is not strictly decreasing, then A_∞ can have unbounded support, hence the right-tails may be non-trivial. Within this chapter two such cases are explored. First, MAPs that satisfy Cramér's condition are considered. In this case a polynomial rate of decay can be determined and, under stronger conditions, an asymptotic expansion can be found. The second case consists of a class of MAPs where the heaviest tailed components of the decomposition (2.3.2) have strong subexponential tails.

5.1 Cramér's Condition

Throughout this section we consider a MAP (J, ξ), where J has state space $E = \{+, -\}$, and the corresponding Lamperti-Kiu process, $Y := J \exp(\xi)$.

It can be shown that the integrals A_∞ and B_∞ are the solutions of *random affine equations*. Recall the decomposition the MAP from Section 2.3.1. We will assume $q_+, q_- > 0$, so that $T_2 < \infty$ and notice that T_2 is the return time of J to its initial state, Y_0, and is independent of Y_0. Then, similarly to the result in [6, Section 4.3], by the Markov additive property

$$B_\infty = B_{T_2} + Y_{T_2}\hat{B}_\infty,$$

where $\hat{B}_\infty$ is an identical but independently distributed copy of B_∞, which is independent of B_{T_2} and Y_{T_2}. Notice that $Y_{T_2} > 0$ a.s. and is independent of J_0 because of its symmetry in the components of the decomposition of Y. Similarly,

$$A_\infty = A_{T_2} + Y_{T_2}\hat{A}_\infty,$$

where $\hat{A}_\infty$ is an independent but identically distributed copy of A_∞ and is independent of Y_{T_2} and A_{T_2}.

Recall $K := \mathbb{E}\left[\xi_{T_2}\right]/\mathbb{E}[T_2]$ from (3.2.1). The following results are generalisations of [6, pp 201, Corollary 5] to Lamperti-Kiu processes using the *implicit renewal theorems* given in [27, Theorem 4.1, pp 135] and [31, Theorem 5, pp 246].

Theorem 5.1.1 (Right Tails in Cramér's Case)
Suppose Y is an infinite lifetime Lamperti-Kiu process with $K < 0$ and there is a $\kappa > 0$ such that $F(\kappa)$ exists, $\lambda(\kappa) = 0$ and

$$\mathbb{E}[|Y_{T_2}|^\kappa \log^+ |Y_{T_2}|] < \infty. \tag{5.1.1}$$

If Y_{T_2} does not have a lattice distribution, then there exist constants $\mathbf{c}_A, \mathbf{c}_B^+, \mathbf{c}_B^- \in \mathbb{R}^{|E|}$ such that,

$$\mathbf{c}_A := \lim_{t\to\infty} t^\kappa \mathbf{P}(A_\infty > t), \quad \mathbf{c}_B^+ := \lim_{t\to\infty} t^\kappa \mathbf{P}(B_\infty > t) \quad \text{and} \quad \mathbf{c}_B^- := \lim_{t\to\infty} t^\kappa \mathbf{P}(B_\infty < -t),$$

hence A_∞, B_∞ have moments of order $s \in \mathbb{C}^+$ if and only if $0 \le \Re(s) < \kappa$.

Proof
Since the theorem assumes (5.1.1), the result is an immediate consequence of [27, pp 129, Section 2, Theorem 2.3] and [31, pp 246, Section 4, Theorem 5], provided:

$$\mathbb{E}[\log |Y_{T_2}|] < 0, \tag{5.1.2}$$

$$\mathbb{E}[|Y_{T_2}|^\kappa] = 1, \tag{5.1.3}$$

$$0 < \mathbb{E}[A_{T_2}^\kappa] < \infty, \tag{5.1.4}$$

$$0 < \mathbb{E}[|B_{T_2}|^\kappa] < \infty. \tag{5.1.5}$$

We now prove that these equations hold under the conditions of the theorem.

To show equation (5.1.2), we expand $\log(Y_{T_2})$ to get

$$\mathbb{E}\left[\log |Y_{T_2}|\right] = \frac{1}{q_+}\mathbb{E}\left[\xi_1^{(+)}\right] + \frac{1}{q_-}\mathbb{E}\left[\xi_1^{(-)}\right] + \mathbb{E}\left[U_+\right] + \mathbb{E}\left[U_-\right] = \left(\frac{1}{q_+} + \frac{1}{q_-}\right) K,$$

then, since $q_+^{-1} + q_-^{-1} > 0$, equation (5.1.2) follows from the assumption $K < 0$.

Since $\det(F(\kappa)) = (\psi_+(z) - q_+)(\psi_-(z) - q_-) - q_+q_-G_+(z)G_-(z)$ and by the assumption that 0 is an eigenvalue of $F(\kappa)$, we have that

$$1 = \left(\frac{q_+G_+(\kappa)}{\psi_+(\kappa) - q_+}\right)\left(\frac{q_-G_-(\kappa)}{\psi_-(\kappa) - q_-}\right). \tag{5.1.6}$$

Let $\lambda_2(z)$ be the other eigenvalue of $F(z)$. Then, for all real $z \in (0, \kappa)$, by Cramér's condition, $\lambda_2(z) < \lambda(z) < 0$ and so $0 < \lambda_2(z)\lambda(z) = \det(F(z))$. Rearranging this gives $(\psi_+(z)-q_+)(\psi_-(z)-q_-) > 0$, hence, $\psi_\pm(z)-q_\pm$ has no roots in $(0, \kappa)$. Since $\psi_\pm(0)-q_\pm < 0$, by continuity, $\psi_\pm(z) - q_\pm < 0$ for all $z \in (0, \kappa)$. By independence and using (5.1.6), we get

$$\mathbb{E}\left[|Y_{T_2}|^\kappa\right] = \mathbb{E}[\exp(\psi_+(\kappa)\zeta_+)]G_+(\kappa)\mathbb{E}[\exp(\psi_-(\kappa)\zeta_-)]G_-(\kappa) = 1,$$

hence equation (5.1.3) holds.

Using independence and the inequality $(a + b)^x \leq 2^x(a^x + b^x)$ for $a, b, x > 0$, we get

$$\mathbb{E}\left[|A_{T_2}|^\kappa\right] \leq 2^\kappa \left(\mathbb{E}\left[\left(\int_0^{\zeta_1} \exp(\xi_s^{(1)})ds\right)^\kappa\right] + \mathbb{E}\left[\exp\left(\kappa\left(\xi_{\zeta_\pm}^{(1)} + U_\pm\right)\right)\right]\mathbb{E}\left[\left(\int_0^{\zeta_2} \exp(\xi_s^{(2)})ds\right)^\kappa\right]\right).$$

From [6], it is know that $\mathbb{E}\left[\left(\int_0^{\zeta_\alpha} \exp(\xi_s^\alpha)ds\right)^x\right] < \infty$ for all $x \in (0,\infty)$ such that $\psi_\alpha(x)-q_\alpha < 0$ and all $\alpha \in E$. This follows from the fact $\int_0^{\zeta_\alpha} \exp(\xi_s^\alpha)ds$ is the exponential functional of the Lévy process $\xi^{(\alpha)}$, sent to the cemetery state $-\infty$ at an independent exponentially distributed time of rate q_α. Then, since we have already seen that $\psi_\alpha(\kappa) - q_\alpha < 0$, it follows that $\mathbb{E}\left[\left(\int_0^{\zeta_\alpha} \exp(\xi_s^\alpha)ds\right)^\kappa\right] < \infty$.

By the assumption that $F(\kappa)$ exists, we have $\mathbb{E}\left[\exp\left(\kappa U_\pm\right)\right] < \infty$, whilst $\mathbb{E}\left[\exp\left(\kappa\xi_{\zeta_\pm}^{(\pm)}\right)\right] < \infty$ by standard results. Hence, $\mathbb{E}\left[|B_{T_2}|^\kappa\right] \leq \mathbb{E}\left[|A_{T_2}|^\kappa\right] < \infty$ and so equations (5.1.4) and (5.1.5) hold. □

Remark 5.1.1

In the case of a Lévy process, where E is a singleton set, it has been shown in [44] that the constants can be found explicitly by evaluating a *Bernstein-Gamma function*.

Remark 5.1.2

The condition that there is a $\kappa > 0$ such that $F(\kappa)$ exists and $\lambda(\kappa) = 0$ is *Cramér's condition*, as outlined in Section 2.3.2. In the case that G_α is continuous for all $\alpha \in E$, equation (5.1.1) follows automatically from Cramér's condition. Indeed, by continuity, we can pick $\epsilon > 0$ such that $\psi_\alpha(\kappa + \epsilon) - q_\alpha < 0$ and $G_\alpha(\kappa + \epsilon) < \infty$ for all $\alpha \in E$. Then, from the proof of equation (5.1.3) we obtain

$$\mathbb{E}\left[|Y_{T_2}|^{\kappa+\epsilon}\right] = \prod_{\alpha \in E} \left(\frac{q_\alpha G_\alpha(\kappa + \epsilon)}{q_\alpha - \psi_\alpha(\kappa + \epsilon)} \right) < \infty.$$

Since $\log^+(x) < x^\epsilon$ for all $x \geq R$, for some sufficiently large $R > 0$, we have

$$\begin{aligned}\mathbb{E}[|Y_{T_2}|^\kappa \log^+ Y_{T_2}] &= \mathbb{E}[|Y_{T_2}|^\kappa \log^+ Y_{T_2}; |Y_{T_2}| \leq R] + \mathbb{E}[|Y_{T_2}|^\kappa \log^+ Y_{T_2}; |Y_{T_2}| > R] \\ &\leq R^{\kappa+\epsilon} + \mathbb{E}[|Y_{T_2}|^{\kappa+\epsilon}] < \infty,\end{aligned}$$

hence (5.1.1) holds.

5.2 Spectrally Positive MAP

For some spectrally positive cases with a non-zero diffusion coefficient, we can obtain more detailed results on the tails of the density of A_∞, by considering the Mellin transform results of Section 4.2.2. In particular, we will consider a spectrally positive MAP (J, ξ) which satisfies Assumption 4.2.2.

Let $Z := \{z \in \mathbb{C} : \det(F(z)) = 0\}$ be the set of points in $\mathbb{C}$ where F is singular and let $Z^- := Z \cap \mathbb{C}^-$ be the subset of Z containing only the points with negative real part. Let $p_\sigma : \mathbb{R}^+ \to \mathbb{R}^+$ denote the density of A_∞ when $J_0 = \sigma \in \{+, -\}$ and $\mathbf{p} := (p_+, p_-)^T$. Also recall the notation $K_\alpha(s) := \frac{q_\alpha - \psi_\alpha(s)}{s}$, for each $s \in \mathbb{C}$ and $\alpha \in E$, from Section 4.2.

Theorem 5.2.1

Suppose Assumption 4.2.2 holds, that $x \in \mathbb{R}^+$, $Z \cap \mathbb{N} = \emptyset$, $s_1 - s_2 \notin \mathbb{Z}$ and $\det'(F(s_1)) \neq 0$ for all $s_1, s_2 \in Z$. Then,

$$\mathbf{p}(x) \sim \sum_{s \in Z} \sum_{n=-\lfloor s \rfloor \wedge 0}^{\infty} \prod_{k=n}^{1} \left(\frac{-F(s+k)}{s+k} \right)^{-1} R(s) \mathbf{m} x^{-s-n-1} \qquad \text{as } x \to \infty, \tag{5.2.1}$$

where, for $s \in Z$,

$$R(s) = \frac{s^2}{\det'(F(s))}\begin{pmatrix} K_-(s) & q_+G_+(s)/s \\ q_-G_-(s)/s & K_+(s) \end{pmatrix}$$
$$\times \lim_{n\to\infty}\prod_{k=1}^{n}\left(\left(\frac{F(s-k)}{k-s}\right)^{-1}\right)\begin{pmatrix} K_+(-k)^{-s} & 0 \\ 0 & K_-(-k)^{-s} \end{pmatrix}\left(\prod_{k=0}^{n}\left(\frac{F(-k)}{k}\right)^{-1}\right)^{-1},$$

and $\mathbf{m} := \mathbb{E}[A_\infty^{-1}] = F'(0)\mathbf{e} + F(0)\mathbf{E}[\log A_\infty]$. *Moreover, if there exists* $c \in (-b, -a)$ *and* $K \in \mathbb{N}$*, such that*

$$\sup_{\substack{k\in\mathbb{N} \\ k>K}}\sup_{s\in\mathbb{R}}\left\|\left(\frac{-F(c+k+is)}{c+k+is}\right)^{-1}\right\| < x, \tag{5.2.2}$$

then,

$$\mathbf{p}(x) = \sum_{s\in Z}\sum_{n=-\lfloor s\rfloor\wedge 0}^{\infty}\prod_{k=n}^{1}\left(\frac{-F(s+k)}{s+k}\right)^{-1}R(s)x^{-s-n-1}\mathbf{m}. \tag{5.2.3}$$

Remark 5.2.1

By Theorem 5.1.1, the leading order term of the series expansion (5.2.1) is $x^{-\kappa-1}$ and hence, the coefficients $\mathbf{c_A}$ from Theorem 5.1.1 are given by $R(\kappa)\mathbf{m}$. Moreover, this implies that all of the singularities of $\mathcal{M}(s)\mathbf{m}$ are to the right of κ.

Remark 5.2.2

In the definition of $R(s)$, there is a product that is closely related to the definition of $M(s)$. The only difference is that the product in $R(s)$ starts at $k = 1$ rather than 0, because it is the $k = 0$ term that produces the singularity.

Remark 5.2.3

The conditions that $s_1 - s_2 \notin Z$ and $\det'(F(s_1)) \neq 0$, for all $s_1, s_2 \in Z$, ensures that all the singularities of $\mathcal{M}$ are simple. Asymptotics may be obtained in the case that these conditions are relaxed, but the expression $R(s)$ will become increasingly complex.

Proof of Theorem 5.2.1

Let $\mathcal{S}_{-b,-a}$ be the strip of $\mathbb{C}^-$ considered in the proof of Proposition 4.2.2 and suppose that $c_0 \in (-b, -a)$. Moreover, let $\mathcal{M}$ be the matrix valued function from Proposition 4.2.2.

By the bounds from Lemma 4.2.6 combined with Lemma 4.1.8, we have that, for all $t > 0$ there exists $T > 0$, such that, $\|\mathcal{M}(s)\| \leq C_1 \exp\left(\left(t - \frac{1}{2}\right)\pi|\Im(s)|\right)$ for all $s \in \mathcal{S}_{-b,-a}$ with

$|\Im(s)| > T$. Hence, $\mathcal{M}(c_0 + iu) \in L^1(\mathbb{R})$ as a function of u and we can apply the *Mellin inversion theorem*, to obtain

$$\mathbf{p}(x) = \frac{1}{2\pi i}\int_{c_0-i\infty}^{c_0+i\infty} x^{-s}\mathcal{M}(s-1)\mathbf{m}\,ds = \frac{1}{2\pi i}\int_{c_0-1+i\infty}^{c_0-1+i\infty} x^{-(s+1)}\mathcal{M}(s)\mathbf{m}\,ds.$$

We wish to use *Cauchy's residue theorem* to evaluate the integral, hence we compute the singularities of $\mathcal{M}$ and the corresponding residues. By considering the matrix product for $\mathcal{M}$ from Proposition 4.2.2, the set of all singularities of $\mathcal{M}$ is given by

$$Q := \{s + n \mid s \in \mathbb{C}, n \in \mathbb{N}_0, \det(F(s)) = 0\}.$$

We also define the subsets $Q_+ := Q \cap \mathbb{C}^+$, $Q_- := Q \cap \mathbb{C}^-$ and $Q_{x,y} := Q \cap \mathcal{S}_{x,y}$ for all $x, y \in \mathbb{R}$.

Consider a singularity, $\bar{s} \in Z$, of F. We first calculate the residue of $(-F(s)/s)^{-1}$ at $\bar{s}$. By a standard matrix inversion,

$$\left(\frac{-F(s)}{s}\right)^{-1} = \frac{s^2}{\det(F(s))}\begin{pmatrix} K_-(s) & q_+G_+(s)/s \\ q_-G_-(s)/s & K_+(s) \end{pmatrix},$$

and by assumption, each of the components of the last matrix is analytic. The assumption $\det'(F(\bar{s})) \neq 0$ ensures that $\bar{s}$ is a simple pole. Thus, by standard results

$$\operatorname{Res}\left(\frac{-F(s)}{s}, \bar{s}\right) = \frac{\bar{s}^2}{\det'(F(\bar{s}))}\begin{pmatrix} K_-(\bar{s}) & q_+G_+(\bar{s})/\bar{s} \\ q_-G_-(\bar{s})/\bar{s} & K_+(\bar{s}) \end{pmatrix}.$$

The assumption that $s_1 - s_2 \notin \mathbb{Z}$, for all $s_1, s_2 \in Z$, ensures that $\bar{s}$ is also a simple pole of $\mathcal{M}$. Hence, the residue of $\mathcal{M}$ at $\bar{s} \in Z$ is given by

$$\begin{aligned}
\operatorname{Res}(\mathcal{M}, \bar{s}) &= \lim_{s\to\bar{s}}(s-\bar{s})\mathcal{M}(s) \\
&= \lim_{s\to\bar{s}}(s-\bar{s}) \lim_{n\to\infty}\left(\prod_{k=0}^{n}\left(\frac{F(s-k)}{k-s}\right)^{-1}\right) \operatorname{diag}\left(\left\{K_\alpha(-n-1)^{-s-1}\right\}_{\alpha\in E}\right) \prod_{k=n+1}^{1}\left(\frac{F(-k)}{k}\right) \\
&= \operatorname{Res}\left(\left(\frac{-F(s)}{s}\right)^{-1}, \bar{s}\right) \lim_{n\to\infty}\prod_{k=1}^{n}\left(\left(\frac{F(\bar{s}-k)}{k-\bar{s}}\right)^{-1}\right) \\
&\qquad \times \operatorname{diag}\left(\left\{K_\alpha(-n-1)^{-\bar{s}-1}\right\}_{\alpha\in E}\right) \prod_{k=n+1}^{1}\left(\frac{F(-k)}{k}\right).
\end{aligned}$$

Since for all $s_1, s_2 \in Z$ we have $s_1 - s_2 \notin \mathbb{Z}$, we can also calculate the residual of $\mathcal{M}$ at $\bar{s}+m$

for all $m \in \mathbb{N}$, to obtain

$$\begin{aligned}\operatorname{Res}(\mathcal{M}, \bar{s}+m) &= \lim_{s\to\bar{s}+m}(s-\bar{s}-m)\mathcal{M}(s) = \lim_{s\to\bar{s}}(s-\bar{s})\mathcal{M}(s+m)\\ &= \lim_{s\to\bar{s}}(s-\bar{s})\prod_{k=m}^{1}\left(\frac{-F(s+k)}{s+k}\right)^{-1}\mathcal{M}(s) = \left(\prod_{k=1}^{m}\frac{-F(\bar{s}+k)}{\bar{s}+k}\right)^{-1}\operatorname{Res}(\mathcal{M}, \bar{s}).\end{aligned}$$

Now suppose $c > c_0$. Then, by Cauchy's residue theorem, we have

$$\begin{aligned}\mathbf{p}(x) &= \sum_{c\in Q_{c_0,c}} \operatorname{Res}(\mathcal{M}\mathbf{m}, s)x^{-s-1} + \mathbf{q}_c(x)\\ &= \sum_{s\in Z_{-\infty,c}}\ \sum_{\substack{n\in\mathbb{N}_0\\ n+s\in\mathcal{S}_{c_0,c}}} \operatorname{Res}(\mathcal{M}\mathbf{m}, s+n)x^{-s-n-1} + \mathbf{q}_c(x),\end{aligned}$$

where $Z_{-\infty,c} \coloneqq \{z \in Z \mid \Re(z) \in (-\infty, c)\}$ and

$$\mathbf{q}_c(x) \coloneqq \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} x^{-s-1}\mathcal{M}(s)\mathbf{m}\, ds.$$

We now consider the behaviour of $\mathbf{q}_c(x)$ as $c \to \infty$. Using the recurrence relation, for any $n \in \mathbb{N}$,

$$\mathbf{q}_c(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} x^{-s-1}\prod_{k=n-1}^{0}\left(\frac{-F(s-k)}{s-k}\right)^{-1}\mathcal{M}(s-n)\mathbf{m}\, ds.$$

Now, let $N_c \in \mathbb{N}$ such that $< c > \coloneqq c - N_c \in (-b, -a)$. Then, we have

$$\mathbf{q}_c(x) = \frac{1}{2\pi i}x^{-N_c-1}\int_{<c>-i\infty}^{<c>+i\infty} x^{-s}\prod_{k=1}^{N_c}\left(\frac{-F(s+k)}{s+k}\right)^{-1}\mathcal{M}(s)\mathbf{m}\, ds.$$

By taking norms, it follows that

$$\|\mathbf{q}_c(x)\| \leq \frac{1}{2\pi}x^{-c-1}\prod_{k=1}^{N_c}\sup_{s\in\mathbb{R}}\left\|\left(\frac{-F(<c>+k+is)}{<c>+k+is}\right)^{-1}\right\|\|\mathbf{m}\|\int_{<c>-i\infty}^{<c>+i\infty}\|\mathcal{M}(s)\|\, ds.$$

Now consider Z, the set of zeros of $\det(F(s))$. Since the components of $F(s)$ are analytic, so is $\det(F(s))$, hence it must have at most countably many zeros. Therefore, the set of $x \in (-b, -a)$ such that there exists an $n \in \mathbb{Z}$ and an $r \in \mathbb{R}$ with $\det(F(x + n + ir)) = 0$ is also countable. Hence, we can choose a $c_0 \in (-b, -a)$ that doesn't coincide with any such x.

Consider $k \in \mathbb{N}$, then by construction, $\det(F(c_0 + k + ir)) \neq 0$ for all $r \in \mathbb{R}$. Moreover, as $r \to \infty$, the diagonal terms of $F(c_0+k+ir)/(c_0+k+ir)$ grow linearly whilst the off-diagonal terms converge to zero since

$$\left|\frac{G_\sigma(a+ir)}{a+ir}\right| = \left|\frac{\mathbb{E}\left[U_\sigma^{a+ir}\right]}{a+ir}\right| \leq \frac{\mathbb{E}\left[U_\sigma^{a}\right]}{|r|} \to 0,$$

as $r \to \pm\infty$. Thus,

$$\lim_{r\to\pm\infty}\left\|\left(\frac{-F(<c>+k+ir}{<c>+k+ir}\right)^{-1}\right\| = 0.$$

Then, $\|(F(s)/s)^{-1}\|$ is bounded on $c_0 + k + i\mathbb{R}$, since $F(s)/s$ is analytic and zero-free on $c_0 + k + i\mathbb{R}$. For $c > 0$ such that $<c>= c_0$, we have such a bound for each $k \in \mathbb{N} \cap [0, N_c]$, hence

$$C_c := \prod_{k=0}^{N_c} \sup_{s\in\mathbb{R}} \left\|\left(\frac{-F(<c>+k+is))}{<c>+k+is}\right)^{-1}\right\| < \infty.$$

This gives the bound

$$\|\mathbf{q}_c(x)\| \le |x|^{-c-1}\frac{\|\mathbf{m}\|}{2\pi}C_c\int_{<c>-i\infty}^{<c>+i\infty}\|\mathcal{M}(s)\|ds.$$

Using the exponential decay of $\|\mathcal{M}(<c>+is)\|$ as $s \to \pm\infty$, the integral converges and hence $\|\mathbf{q}_c(x)\| = \mathcal{O}(x^{-c-1})$ as $x \to \infty$. Hence, taking $c \to \infty$ yields the series expansion (5.2.1).

We now consider the case where the Mellin inversion can be performed. Fix $x \in \mathbb{R}^+$ and suppose there exists a $c_0 \in (-b, -a)$ such that condition (5.2.2) holds. Then, let

$$\gamma := \sup_{\substack{k\in\mathbb{N}\\ k>K}} \sup_{s\in\mathbb{R}} \left\|\left(\frac{-F(c+k+is)}{c+k+is}\right)^{-1}\right\| < x,$$

thus,

$$\|\mathbf{q}_c(x)\| \le |x|^{-c-1}\gamma^{N_c}\frac{\|\mathbf{m}\|}{2\pi}\int_{<c>-i\infty}^{<c>+i\infty}\|\mathcal{M}(s)\|ds.$$

Then, since $\lim_{c\to\infty} x^{-c-1}\gamma^{N_c} = 0$, we have that $\lim_{c\to 0}\|\mathbf{q}_c(x)\| = 0$, from which (5.2.3) follows. $\square$

5.3 Strong Subexponential Tails

When not all of the conditions of Theorem 5.1.1 hold, a different approach to the investigation of the tails and moments of A_∞ is required. In Theorem 5.1.1, it is assumed that $F(\kappa)$ exists for some $\kappa > 0$, which requires that positive exponential moments of ξ must exist. This is a condition that does not necessarily hold in general.

In this section, we consider the case that these conditions fail to hold because some of the components of the decomposition (2.3.2) are *strong subexponential*. We establish the tail asymptotics of the distribution of A_∞ in a generalisation of the corresponding result for a Lévy process, considered in [39, pp 166, Section 4]. Interestingly, the resulting tails are of a very different nature to those considered under Cramér's condition.

5.3.1 Introduction to Types of Heavy Tailed Distributions

Strong subexponential distributions are a widely studied class of heavy tailed distributions, both because of their mathematical tractability and their appearance in empirical data (for example see [14] and [29]). We will use [24] as a reference to the background theory of subexponential distributions, within which further discussion of the use of these distributions can be found. In particular, we will use the following definitions from [24].

Definition 5.3.1 (Long Tailed and (Strong) Subexponential Distributions)
Let $Q : \mathbb{R} \to [0, 1]$ be a probability distribution and $\bar{Q}(x) := 1 - Q(x)$ for all $x \in \mathbb{R}$. Then, we make the following definitions:

(i) Q is a *long-tailed distribution* if $Q(x + y)/Q(x) \to 1$ as $x \to \infty$ and for any $y \in \mathbb{R}^+$.

(ii) If $\overline{Q * Q}(x)/\overline{Q}(x) \to 2$ as $x \to \infty$ and Q is long tailed, then Q is a *subexponential distribution*.

(iii) Q is *strong subexponential* if Q is long tailed and

$$\lim_{z\to\infty} \frac{1}{\bar{Q}(z)} \int_0^z \bar{Q}(z - x)\bar{Q}(x)dx = 2m,$$

where $m := \mathbb{E}[X]$ for a random variable X with distribution Q.

It can be shown (for instance see [24]) that all strong subexponential distributions are subexponential and by definition all subexponential distributions are long-tailed. If a random variable has a heavy tailed or (strong) subexponential distribution, then we will refer to it as a heavy tailed or (strong) subexponential random variable, respectively.

Let $\mathcal{S}$ denote the set of real valued *subexponential random variables* and $\mathcal{S}^*$ denote the subset of $\mathcal{S}$ comprising of *strong subexponential random variables*. For any two functions $f, g : \mathbb{R} \to \mathbb{R}$, we will write $f \sim g$ if $\lim_{x\to\infty} f(x)/g(x) = 1$, $f = \mathcal{O}(g)$ if $\limsup_{x\to\infty} f(x)/g(x) \in \mathbb{R}$ and $f = o(g)$ if $\lim_{x\to\infty} f(x)/g(x) = 0$.

For a random variable X, define the functions

$$G_X(x) := \int_x^\infty \mathbb{P}(X > u)\, du \qquad \text{and} \qquad H_X(x) := \min(1, G_X(x)), \tag{5.3.1}$$

where H_X is referred to as the *integrated tail* of X. If H_X is a subexponential distribution, then we write $X \in \mathcal{S}_I$ and from [24, Chapter 3, pp 55, Theorem 3.27], it is known that $\mathcal{S}^* \subset \mathcal{S}_I$.

5.3.2 Framework for an Upper Bound of $\log(A_\infty)$

We now develop a framework for bounding $\log(A_\infty)$ whenever $\mathbb{E}[\xi_{T_2}] \in (-\infty, 0)$. This will be used in the strong subexponential setting of Section 5.3.3 to obtain the right tails of A_∞.

Recall $K = \mathbb{E}[\xi_{T_2}]/\mathbb{E}[T_2]$ from (3.2.1). For $\epsilon \in (0, -K)$ and sufficiently large $A \in \mathbb{R}$, define a sequence of stopping times by $\sigma_0 := 0$ and, for each $n \in \mathbb{N}$,

$$\sigma_n := \inf\left\{t > \sigma_{n-1} \,:\, \xi_t - \xi_{\sigma_{n-1}} \geq (K+\epsilon)(t - \sigma_{n-1}) + A\right\}, \tag{5.3.2}$$

with the convention $\inf(\varnothing) = \infty$, and setting $\sigma_n = \infty$ if $\sigma_{n-1} = \infty$. Then, also define $N := \max\{k \in \mathbb{N}_0 \mid \sigma_k < \infty\}$ and, for each $n \in \mathbb{N}$, let $\rho_n := \mathbb{P}(\sigma_n < \infty \mid \sigma_{n-1} < \infty)$. The following lemma concerns the finiteness of N.

Lemma 5.3.1

If $\mathbb{E}[\xi_{T_2}] \in (-\infty, 0)$, there exists an $A^* > 0$ such that, for all $A > A^*$, N is a.s. finite.

Proof

Define a new MAP, $\{(J_t, \tilde{\xi}_t) \,:\, t \geq 0\}$, by setting $\tilde{\xi}_t := \xi_t - (K+\epsilon)t$. Then, for each $n \in \mathbb{N}$,

$$1 - \rho_n = \mathbb{P}\left(\sup_{t > \sigma_{n-1}} \tilde{\xi}_t - \tilde{\xi}_{\sigma_{n-1}} < A \,\middle|\, \sigma_{n-1} < \infty\right).$$

Since σ_{n-1} is a stopping time, using the Markov additive property and summing over the events $\{J_{\sigma_{n-1}} = \alpha\}$ for $\alpha \in \{+,-\}$ gives

$$1 - \rho_n = \sum_{\alpha \in \{+,-\}} \mathbb{P}_\alpha\left(\sup_{t \geq 0} \tilde{\xi}_t < A\right) \mathbb{P}(J_{\sigma_{n-1}} = \alpha).$$

By the strong law of large numbers,

$$\lim_{t\to\infty} t^{-1}\tilde{\xi}_t = \lim_{t\to\infty} t^{-1}(\xi_t - (K+\epsilon)t) = K - (K+\epsilon) = -\epsilon < 0 \qquad \text{a.s..}$$

Hence, there a.s. exists $T > 0$ such that if $t > T$, then $\tilde{\xi}_t < 0$. It follows that $\sup_{t\geq 0} \tilde{\xi}_t = \max\left(0\,,\, \sup_{t \in [0,T]} \tilde{\xi}_t\right) < \infty$ a.s., since the supremum of a càdlàg process over a compact interval is bounded.

This implies that there exists an $A^* > 0$, such that $\mathbb{P}\left(\sup_{t\geq 0} \tilde{\xi}_t > A\right) < 1$, for all $A > A^*$. From this, we conclude $N < \infty$ a.s., since

$$\mathbb{P}(N > n) = \prod_{k=1}^{n} \mathbb{P}(\sigma_k < \infty \mid \sigma_{k-1} < \infty) \leq \max_{\alpha \in \{+,-\}} \mathbb{P}_\alpha\left(\sup_{t\geq 0} \tilde{\xi}_t < A\right)^n. \tag{5.3.3}$$

$\square$

Since there are conditions for N to be finite, we can bound $\log(A_\infty)$ from above by using the stopping times $\{\sigma_n\}_{n\in\mathbb{N}}$ to split the process $\{\xi_t : t \geq 0\}$ into a finite number of sections, each bounded from above. Now define the constant

$$C := \log\left(\frac{e^A}{|K+\epsilon|}\right),$$

where A is sufficiently large to ensure $e^C > 2$. The following lemma provides an upper bound for $\log(A_\infty)$.

Lemma 5.3.2

If $\mathbb{E}[\xi_{T_2}] \in (-\infty, 0)$, *then*

$$\log A_\infty \leq (N+1)C + \sum_{n=1}^{N} (\xi_{\sigma_n} - \xi_{\sigma_{n-1}})^+,$$

where $(\cdot)^+$ *denotes the positive part.*

Proof

Following the approach of [39, pp 11, Lemma 4.1], A_∞ may be expanded as

$$A_\infty = \int_0^{\sigma_1} e^{\xi_t} dt + e^{\xi_{\sigma_1}} \left(\int_{\sigma_1}^{\sigma_2} e^{\xi_t - \xi_{\sigma_1}} dt + e^{\xi_{\sigma_2} - \xi_{\sigma_1}} \left(\int_{\sigma_2}^{\sigma_3} e^{\xi_t - \xi_{\sigma_2}} dt + \dots \right.\right.$$
$$\left.\left. \dots + e^{\xi_{\sigma_N} - \xi_{\sigma_{N-1}}} \left(\int_{\sigma_N}^{\sigma_{N+1}} e^{\xi_t - \xi_{\sigma_N}} dt \right) \right) \right),$$

noting that $\sigma_{N+1} = \infty$. Then, for each $n \leq N$, by the definition of σ_n,

$$\int_{\sigma_n}^{\sigma_{n+1}} e^{\xi_t - \xi_{\sigma_n}}\, dt \leq \int_{\sigma_n}^{\sigma_{n+1}} \exp((K+\epsilon)(t-\sigma_n) + A)\, dt \leq e^C,$$

which substituted into the expression for A_∞ gives

$$A_\infty \leq e^C + e^{\xi_{\sigma_1}} \left(e^C + e^{\xi_{\sigma_2} - \xi_{\sigma_1}} \left(e^C + \dots + e^{\xi_{\sigma_N} - \xi_{\sigma_{N-1}}} \left(e^C \right) \right) \right).$$

Then, considering the logarithm of both sides and repeatedly using the property $\log(A+B) \leq \log(A) + \log(B)$ whenever $A, B > 2$,

$$\log(A_\infty) \leq (N+1)C + \sum_{n=1}^{N} (\xi_{\sigma_n} - \xi_{\sigma_{n-1}})^+,$$

where the log inequality could be used since $e^{(\xi_{\sigma_n} - \xi_{\sigma_{n-1}})^+} \geq 1$ and $e^C > 2$. $\square$

As a consequence of this lemma, the right tails of $\log(A_\infty)$ can be studied by considering the evolution of the MAP between the stopping times $\{\sigma_n\}_{n\in\mathbb{N}}$. First we consider the J component.

Let $\{M_n\}_{n\in\mathbb{N}_0}$ be the sequence of random variables, taking values in $\{+,-,\infty\}$, such that for each $n \in \mathbb{N}$, if $\sigma_n < \infty$ then $M_n = J_{\sigma_n}$ otherwise $M_n = \infty$. For each $\alpha, \beta \in \{+,-\}$, we will be interested in the number of times that $\{M_n\}_{n\in\mathbb{N}}$ transitions from α to β. For this purpose, define the random variable $N(\alpha,\beta) := \sum_{k=1}^{\infty} \mathbb{1}_{\{M_{k-1}=\alpha, M_k=\beta\}}$. Then, some properties of $\{M_n\}_{n\in\mathbb{N}}$ are summarised in the following proposition.

Proposition 5.3.1

Suppose $\mathbb{E}[Y_{T_2}] \in (-\infty, 0)$. Then, the sequence $\{M_n\}_{n\in\mathbb{N}_0}$ is a discrete time homogeneous Markov chain, with ∞ as an absorbing state. Moreover, if η is the stochastic matrix of $\{M_n\}_{n\in\mathbb{N}}$ and $\alpha, \beta, \gamma \in \{+,-\}$ such that $\alpha \neq \beta$, then

$$\eta_{\alpha,\beta} \to 0, \qquad \mathbb{E}_\alpha[N(\alpha,\gamma)] \sim \eta_{\alpha,\gamma} \qquad \textit{and} \qquad \mathbb{E}_\beta[N(\alpha,\gamma)] = o(\eta_{\alpha,\gamma}),$$

as $A \to \infty$.

Proof

First, we show that $\{M_n\}_{n\in\mathbb{N}}$ is a Markov chain. If $M_{n-1} \neq \infty$, then by the Markov additive property, since σ_{n-1} is a stopping time, $\{\xi_{\sigma_{n-1}+t} - \xi_{\sigma_{n-1}} : t \geq 0\}$ is independent of $\mathcal{F}_{\sigma_{n-1}}$, given M_{n-1}. Moreover, the random variable

$$\Delta\sigma_n := \sigma_n - \sigma_{n-1} = \inf\{t \geq 0 : \xi_{t+\sigma_{n-1}} - \xi_{\sigma_{n-1}} \geq t(K+\epsilon) + A\},$$

is a function of $\{\xi_{t+\sigma_{n-1}} - \xi_{\sigma_{n-1}} : t \geq 0\}$. Thus, the event $\{M_n = \infty\} = \{\Delta\sigma_n = \infty\}$ is independent of $\mathcal{F}_{\sigma_{n-1}}$ given M_{n-1} and has the same law as the event $\{M_1 = \infty\}$ given K_0.

If $\Delta\sigma_n < \infty$, then $M_n = J_{\sigma_n} = J_{\sigma_{n-1}+\Delta\sigma_n}$, hence M_n is a function of $\left\{\left(\xi_{\sigma_{n-1}+t} - \xi_{\sigma_{n-1}}, J_{\sigma_{n-1}+t}\right) : t \geq 0\right\}$. Then, by the Markov additive property, M_n is independent of $\mathcal{F}_{\sigma_{n-1}}$ given M_{n-1} and has the same distribution as M_1 given M_0. Hence, the sequence $\{M_n\}_{n\in\mathbb{N}}$ is a time homogeneous Markov chain. By definition of σ_n, ∞ is clearly an absorbing state for $\{M_n\}_{n\in\mathbb{N}}$.

We now consider the limiting behaviour of η as $A \to \infty$. Let $\alpha, \beta \in \{+,-\}$. From the proof of Lemma 5.3.1, we know that $\sup_{t\geq 0} \tilde{\xi}_t < \infty$ a.s., where $\tilde{\xi}_t := \xi_t - (K+\epsilon)t$. Thus,

$$\lim_{A\to\infty} \mathbb{P}\left(\sup_{t\geq 0} \{\xi_t - (K+\epsilon)t\} > A\right) = \lim_{A\to\infty} \mathbb{P}\left(\sup_{t\geq 0} \tilde{\xi}_t > A\right) = 0,$$

however,

$$\eta_{\alpha,+} + \eta_{\alpha,-} = \mathbb{P}_\alpha(\sigma_1 < \infty) = \mathbb{P}_\alpha\left(\sup_{t\geq 0} \{\xi_t - (K+\epsilon)t\} > A\right).$$

Since $\eta_{\alpha,\beta}$ is non-negative, this implies $\lim_{A\to\infty}\eta_{\alpha,\beta}=0$.

Further, assume that $\gamma\in\{+,-\}$ and $\alpha\neq\beta$. Then, it is easily seen that

$$\mathbb{E}_\sigma[N(\alpha,\gamma)]=\sum_{n=1}^{\infty}\mathbb{P}_\sigma(M_n=\gamma\mid M_{n-1}=\alpha)\mathbb{P}_\sigma(M_{n-1}=\alpha)=\eta_{\alpha,\gamma}\left(\mathbb{1}_{\{\sigma=\alpha\}}+\phi_\sigma(\alpha)\right),$$

where $\phi_\sigma(\alpha):=\sum_{n=1}^{\infty}\mathbb{P}_\sigma(M_n=\alpha)$. Since ∞ is an absorbing state of the Markov chain $\{M_n\}_{n\in\mathbb{N}}$ and $\alpha\neq\beta$, for each $n\in\mathbb{N}$,

$$\begin{aligned}\mathbb{P}_\sigma(M_n=\alpha)&=\mathbb{P}_\sigma(M_n=\alpha\mid M_{n-1}=\alpha)\mathbb{P}_\sigma(M_{n-1}=\alpha)\\&\qquad+\mathbb{P}_\sigma(M_n=\alpha\mid M_{n-1}=\beta)\mathbb{P}_\sigma(M_{n-1}=\beta)\\&=\eta_{\alpha,\alpha}\mathbb{P}_\sigma(M_{n-1}=\alpha)+\eta_{\beta,\alpha}\mathbb{P}_\sigma(M_{n-1}=\beta).\end{aligned}$$

Then, summing up over $n\in\mathbb{N}$ yields

$$\phi_\sigma(\alpha)=\eta_{\alpha,\alpha}\left(\mathbb{1}_{\{\sigma=\alpha\}}+\phi_\sigma(\alpha)\right)+\eta_{\beta,\alpha}\left(\mathbb{1}_{\{\sigma=\beta\}}+\phi_\sigma(\beta)\right),$$

and by symmetry

$$\phi_\sigma(\beta)=\eta_{\beta,\beta}\left(\mathbb{1}_{\{\sigma=\beta\}}+\phi_\sigma(\beta)\right)+\eta_{\alpha,\beta}\left(\mathbb{1}_{\{\sigma=\alpha\}}+\phi_\sigma(\alpha)\right).$$

Solving this system gives

$$\phi_\sigma(\alpha)=\frac{\mathbb{1}_{\{\sigma=\alpha\}}\left(\eta_{\alpha,\alpha}(1-\eta_{\beta,\beta})+\eta_{\beta,\alpha}\eta_{\alpha,\beta}\right)+\mathbb{1}_{\{\sigma=\beta\}}\eta_{\beta,\alpha}}{(1-\eta_{\alpha,\alpha})(1-\eta_{\beta,\beta})-\eta_{\beta,\alpha}\eta_{\alpha,\beta}},$$

thus,

$$\mathbb{E}_\sigma\left[N(\alpha,\gamma)\right]=\frac{\eta_{\alpha,\gamma}\left(\mathbb{1}_{\{\sigma=\alpha\}}(1-\eta_{\beta,\beta})+\mathbb{1}_{\{\sigma=\beta\}}\eta_{\beta,\alpha}\right)}{(1-\eta_{\alpha,\alpha})(1-\eta_{\beta,\beta})-\eta_{\beta,\alpha}\eta_{\alpha,\beta}},$$

from which the result of the lemma is immediate. The asymptotic results then follow from the limiting behaviour of η. □

In the next lemma we consider the evolution of ξ between the stopping times $\{\sigma_n\}_{n\in\mathbb{N}}$, by conditioning on the values of $\{M_n\}_{n\in\mathbb{N}}$.

Lemma 5.3.3

Suppose $\mathbb{E}[\xi_{T_2}]\in(-\infty,0)$ and that $m,n\in\mathbb{N}$ with $m<n$. Then, conditionally on M_{m-1}, M_m, M_{n-1} and M_n, the increments $\xi_{\sigma_m}-\xi_{\sigma_{m-1}}$ and $\xi_{\sigma_n}-\xi_{\sigma_{n-1}}$ are independent. If $\alpha,\beta\in\{+,-\}$, then conditional on the event $\{M_{n-1}=M_{m-1}=\alpha;M_n=M_m=\beta\}$ the increments $\xi_{\sigma_m}-\xi_{\sigma_{m-1}}$ and $\xi_{\sigma_n}-\xi_{\sigma_{n-1}}$ are equal in distribution and independent. Furthermore, for any $l\in\mathbb{N}$ such that $m\neq l$ and any bounded continuous function $f:\mathbb{R}\to\mathbb{R}^+$,

$$\mathbb{E}[f(\xi_{\sigma_m}-\xi_{\sigma_{m-1}})\mid\sigma(M_{m-1},M_m,M_{l-1},M_l)]=\mathbb{E}[f(\xi_{\sigma_m}-\xi_{\sigma_{m-1}})\mid\sigma(M_{m-1},M_m)].$$

Proof

First suppose that $m < l$, then

$$\begin{aligned}&\mathbb{E}[f\left(\xi_{\sigma_m} - \xi_{\sigma_{m-1}}\right) \mid \sigma(M_{m-1}, M_m, M_{l-1}, M_l)] \\ &\quad = \sum_{\gamma,\delta\in\{+,-\}} \mathbb{1}_{\{M_{l-1}=\gamma, M_l=\delta\}} \frac{\mathbb{E}[f\left(\xi_{\sigma_m} - \xi_{\sigma_{m-1}}\right); M_{l-1} = \gamma, M_l = \delta \mid \sigma(M_{m-1}, M_m)]}{\mathbb{P}(M_{l-1} = \gamma, M_l = \delta \mid \sigma(M_{m-1}, M_m))}.\end{aligned}$$

By using the tower property and the fact that $\{M_k\}_{k\in\mathbb{N}}$ is a Markov chain, it follows that

$$\begin{aligned}&\mathbb{E}[f\left(\xi_{\sigma_m} - \xi_{\sigma_{m-1}}\right); M_{l-1} = \gamma, M_l = \delta \mid \sigma(M_{m-1}, M_m)] \\ &\quad = \mathbb{E}\left[f\left(\xi_{\sigma_m} - \xi_{\sigma_{m-1}}\right) \mathbb{E}\left[\mathbb{1}_{\{M_{l-1}=\gamma, M_l=\delta\}} \mid \mathcal{F}_{\sigma_m}\right] \mid \sigma(M_{m-1}, M_m)\right] \\ &\quad = \mathbb{E}\left[f\left(\xi_{\sigma_m} - \xi_{\sigma_{m-1}}\right) \mathbb{E}\left[\mathbb{1}_{\{M_{l-1}=\gamma, M_l=\delta\}} \mid \sigma(M_m)\right] \mid \sigma(M_{m-1}, M_m)\right] \\ &\quad = \mathbb{P}\left(M_{l-1} = \gamma, M_l = \delta \mid \sigma(M_{m-1}, M_m)\right) \mathbb{E}\left[f\left(\xi_{\sigma_m} - \xi_{\sigma_{m-1}}\right) \mid \sigma(M_{m-1}, M_m)\right].\end{aligned}$$

Substituting this into the previous equation gives

$$\begin{aligned}&\mathbb{E}[f\left(\xi_{\sigma_m} - \xi_{\sigma_{m-1}}\right) \mid \sigma(M_{m-1}, M_m, M_{l-1}, M_l)] \\ &\quad = \sum_{\gamma,\delta\in\{+,-\}} \mathbb{1}_{\{M_{l-1}=\gamma, M_l=\delta\}} \mathbb{E}\left[f\left(\xi_{\sigma_m} - \xi_{\sigma_{m-1}}\right) \mid \sigma(M_{m-1}, M_m)\right] \\ &\quad = \mathbb{E}\left[f(\xi_{\sigma_m} - \xi_{\sigma_{m-1}}) \mid \sigma(M_{m-1}, M_m)\right].\end{aligned}$$

Now suppose $m > l$, then, through a direct application of the Markov additive property, it follows that

$$\begin{aligned}&\mathbb{E}\left[f(\xi_{\sigma_m} - \xi_{\sigma_{m-1}}) \mid \sigma(M_{l-1}, M_l, M_{m-1}, M_m)\right] \\ &\quad = \sum_{\alpha\in\{+,-\}} \mathbb{1}_{\{M_m=\alpha\}} \frac{\mathbb{E}\left[f(\xi_{\sigma_m} - \xi_{\sigma_{m-1}}); M_m = \alpha \mid \sigma(M_{l-1}, M_l, M_{m-1})\right]}{\mathbb{P}(M_m = \alpha \mid \sigma(M_{l-1}, M_l, M_{m-1}))} \\ &\quad = \sum_{\alpha\in\{+,-\}} \mathbb{1}_{\{M_m=\alpha\}} \frac{\mathbb{E}\left[f(\xi_{\sigma_m} - \xi_{\sigma_{m-1}}); M_m = \alpha \mid \sigma(M_{m-1})\right]}{\mathbb{P}(M_m = \alpha \mid \sigma(M_{m-1}))} \\ &\quad = \mathbb{E}\left[f(\xi_{\sigma_m} - \xi_{\sigma_{m-1}}) \mid \sigma(M_{m-1}, M_m)\right].\end{aligned}$$

To see the independence of increments, suppose that $f, g : \mathbb{R} \to \mathbb{R}^+$ are bounded continuous functions, then

$$\begin{aligned}&\mathbb{E}\left[f(\xi_{\sigma_n} - \xi_{\sigma_{n-1}}) g(\xi_{\sigma_m} - \xi_{\sigma_{m-1}}) \mid \sigma(M_{m-1}, M_m, M_{n-1}, M_n)\right] \\ &\quad = \sum_{\alpha\in\{+,-\}} \mathbb{1}_{\{K_n=\alpha\}} \frac{\mathbb{E}\left[f(\xi_{\sigma_n} - \xi_{\sigma_{n-1}}) g(\xi_{\sigma_m} - \xi_{\sigma_{m-1}}); M_n = \alpha \mid \sigma(M_{m-1}, M_m, M_{n-1})\right]}{\mathbb{P}\left(M_n = \alpha \mid \sigma(M_{m-1}, M_m, M_{n-1})\right)}.\end{aligned}$$

Then, by the tower property,

$$
\begin{aligned}
&\mathbb{E}\left[f(\xi_{\sigma_n}-\xi_{\sigma_{n-1}})g(\xi_{\sigma_m}-\xi_{\sigma_{m-1}}); M_n=\alpha \mid \sigma(M_{m-1},M_m,M_{n-1})\right] \\
&\quad= \mathbb{E}\left[g(\xi_{\sigma_m}-\xi_{\sigma_{m-1}})\mathbb{E}\left[f(\xi_{\sigma_n}-\xi_{\sigma_{n-1}}); M_n=\alpha \mid \mathcal{F}_{\sigma_{n-1}}\right] \mid \sigma(M_{m-1},M_m,M_{n-1})\right] \\
&\quad= \mathbb{E}\left[g(\xi_{\sigma_m}-\xi_{\sigma_{m-1}})\mathbb{E}\left[f(\xi_{\sigma_n}-\xi_{\sigma_{n-1}}); M_n=\alpha \mid \sigma(M_{\sigma_{n-1}})\right] \mid \sigma(M_{m-1},M_m,M_{n-1})\right] \\
&\quad= \mathbb{E}\left[f(\xi_{\sigma_n}-\xi_{\sigma_{n-1}}); M_n=\alpha \mid \sigma(M_{\sigma_{n-1}})\right]\mathbb{E}\left[g(\xi_{\sigma_m}-\xi_{\sigma_{m-1}}) \mid \sigma(M_{m-1},M_m)\right].
\end{aligned}
$$

Plugging this into the previous equation yields

$$
\begin{aligned}
&\mathbb{E}\left[f(\xi_{\sigma_n}-\xi_{\sigma_{n-1}})g(\xi_{\sigma_m}-\xi_{\sigma_{m-1}}) \mid \sigma(M_{m-1},M_m,M_{n-1},M_n)\right] \\
&= \sum_{\alpha\in\{+,-\}} \mathbb{1}_{\{M_n=\alpha\}} \frac{\mathbb{E}\left[f(\xi_{\sigma_n}-\xi_{\sigma_{n-1}}); M_n=\alpha \mid \sigma(M_{\sigma_{n-1}})\right]\mathbb{E}\left[g(\xi_{\sigma_m}-\xi_{\sigma_{m-1}}) \mid \sigma(M_{m-1},M_m)\right]}{\mathbb{P}\left(M_n=\alpha \mid \sigma(M_{m-1},M_m,M_{n-1})\right)} \\
&= \mathbb{E}\left[g(\xi_{\sigma_m}-\xi_{\sigma_{m-1}}) \mid \sigma(M_{m-1},M_m)\right] \sum_{\alpha\in\{+,-\}} \mathbb{1}_{\{K_n=\alpha\}} \frac{\mathbb{E}\left[f(\xi_{\sigma_n}-\xi_{\sigma_{n-1}}); M_n=\alpha \mid \sigma(M_{\sigma_{n-1}})\right]}{\mathbb{P}\left(M_n=\alpha \mid \sigma(M_{n-1})\right)} \\
&= \mathbb{E}\left[g(\xi_{\sigma_m}-\xi_{\sigma_{m-1}}) \mid \sigma(M_{m-1},M_m)\right]\mathbb{E}\left[f(\xi_{\sigma_n}-\xi_{\sigma_{n-1}}) \mid \sigma(M_{\sigma_{n-1}},M_{\sigma_n})\right].
\end{aligned}
$$

□

5.3.3 MAPs of Strong Subexponential Type

For a random variable X, recall the definition of the integrated tail, H_X, from (5.3.1). For ease of notation, define $H(x) := H_{\xi_{T_2}}(x)$ and, for each $\alpha \in \{+,-\}$, let $H_{\xi_\alpha} := H_{\xi^{(\alpha)}_{\zeta_\alpha}}$ and $H^{(\alpha)} := H_{\xi_\alpha} + H_{U_{-\alpha}}$. Also consider the subset of components of the decomposition (2.3.2) given by $L := \left\{\xi^{(+)}_{\zeta_+}, \xi^{(-)}_{\zeta_-}, U_+, U_-\right\}$.

This section will focus on MAPs where the dominating component of the decomposition is strong subexponential. More precisely, the following definition will be used:

Definition 5.3.2 (MAP of Strong Subexponential Type)
A MAP (J,ξ) is of *strong subexponential type* if ξ_{T_2} is long tailed and there exists $X \in L$ such that $X \in \mathcal{S}^*$ and for all $W \in L \setminus \{X\}$ we have $\mathbb{P}(W > x) = \mathcal{O}(\mathbb{P}(X > x))$.

If a Lamperti-Kiu process is of this type, then there is a heaviest tailed component of the decomposition (2.3.2) and that component is strong subexponential.

For the remainder of this section, the MAP (J,ξ) is assumed to be of strong subexponential type. Let $B \subseteq \{+,-\}$ be the set of all $\beta \in \{+,-\}$, such that $\limsup_{x\to\infty} H_X(x)^{-1}H^{(\beta)}(x) \neq 0$. Then, for any $b \in B$ and $\beta \in \{+,-\} \setminus B$, we have that, $H^{(\beta)}(x) = o\left(H^{(b)}(x)\right)$ as $x \to \infty$.

By the closure properties of $\mathcal{S}$, from [24, pp 52, Chapter 3, Corollary 3.16] it follows that ξ_{T_2} is also strong subexponential. Moreover, the tails of ξ_{T_2} are given by

$$\mathbb{P}\left(\xi_{T_2} > x\right) \sim \sum_{\beta \in \{+,-\}} \left(\mathbb{P}\left(\xi^{(\beta)}_{\zeta_\beta} > x\right) + \mathbb{P}\left(U_\beta > x\right)\right),$$

as $x \to \infty$ and the integrated tails are given by

$$H(x) \sim \sum_{\beta \in \{+,-\}} H^{(\beta)}(x) \sim \sum_{\beta \in B} H^{(\beta)}(x).$$

The main result of this section, which extends [39, Section 4, pp 166] to Lamperti-Kiu processes, is the following theorem.

Theorem 5.3.1 (Right Tails of A_∞ for MAPs of Strong Subexponential Type)
Suppose that (J, ξ) is a MAP of strong subexponential type, such that $\mathbb{E}[\xi_{T_2}] \in (-\infty, 0)$, then

$$\mathbb{P}(A_\infty > x) \sim \frac{H(\log(x))}{|\mathbb{E}[\xi_{T_2}]|}, \qquad \text{as } x \to \infty. \tag{5.3.4}$$

Furthermore, A_∞ is long tailed and $\log(A_\infty)$ is subexponential.

The proof of Theorem 5.3.1 makes use of the framework in Section 5.3.2 and is broken into a number of lemmas.

In the next lemma, we show that the integrated tail of a long-tailed random variable is asymptotically equivalent to an infinite series. This will be used in Lemma 5.3.5 to show the asymptotic equivalence of two distributions.

Lemma 5.3.4
Suppose $K + \epsilon < 0$ and that X is a long tailed random variable, which is independent of $\{T_{2n}\}_{n \in \mathbb{N}}$. Then, as $x \to \infty$,

$$G_X(x) \sim \mathbb{E}[T_2]|K + \epsilon| \sum_{n=0}^{\infty} \mathbb{P}(X > x - (K + \epsilon)T_{2n}), \tag{5.3.5}$$

where G_X is the function defined in (5.3.1) and $K := \mathbb{E}[\xi_{T_2}]/\mathbb{E}[T_2]$ is as defined in (3.2.1).

Proof
Splitting the interval $(0, \infty)$ into a disjoint union gives

$$G_X(x) = \sum_{n=0}^{\infty} \int_{-T_{2n}(K+\epsilon)}^{-T_{2(n+1)}(K+\epsilon)} \mathbb{P}(X > u + x)du.$$

Then, by using the change of variables $u_2 = -(K+\epsilon)^{-1}u_1$,

$$G_X(x) = \sum_{n=0}^{\infty} \int_{T_{2n}}^{T_{2(n+1)}} \mathbb{P}\left(X > -(K+\epsilon)u + x\right) |K+\epsilon| du.$$

By independence of $\{T_{2n}\}_{n\in\mathbb{N}}$ and X, the domain of integration can be shifted to obtain

$$G_X(x) = \sum_{n=0}^{\infty} \int_{0}^{T_{2(n+1)}-T_{2n}} \mathbb{P}\left(X > x - (K+\epsilon)(u+T_{2n}) \,\Big|\, T_{2n}\right) |K+\epsilon|\, du.$$

Taking expectations and noting that the left hand side is not random, that $T_{2(n+1)} - T_{2n} \overset{\mathcal{L}}{=} T_2$ and that $T_{2(n+1)} - T_{2n}$ is independent of T_{2n} gives

$$G_X(x) = |K+\epsilon| \sum_{n=0}^{\infty} \mathbb{E}\left[\int_0^{\tilde{T}_2} \mathbb{P}\left(X > x - (K+\epsilon)(u+T_{2n}) \,\Big|\, T_{2n}\right) du\right],$$

where $\tilde{T}_2$ is an independent and identically distributed copy of T_2. This can be written in the integral form

$$G_X(x) = |K+\epsilon| \sum_{n=0}^{\infty} \int_0^{\infty} \mathbb{P}(\tilde{T}_2 \in ds) \int_0^s \int_0^{\infty} \mathbb{P}(T_{2n} \in dv) \mathbb{P}\left(X > x - (K+\epsilon)(u+v)\right) du. \tag{5.3.6}$$

Let $\delta > 0$. Since X is long-tailed, for all $s > 0$ there exists an $R(s) > 0$ such that, whenever $z > R(s)$ and $y \in [0, -s(K+\epsilon)]$,

$$(1-\delta) \leq \frac{\mathbb{P}(X > z+y)}{\mathbb{P}(X > z)} \leq (1+\delta).$$

Since $-v(K+\epsilon) \geq 0$ for $v \geq 0$ we have, for all $x > R(s)$ and $u \in [0, s]$,

$$(1-\delta) \leq \frac{\mathbb{P}(X > x - (K+\epsilon)(v+u))}{\mathbb{P}(X > x - (K+\epsilon)v)} \leq (1+\delta).$$

To show the lower bound, use this inequality within the last two integrals of (5.3.6) to obtain, for $x > R(s)$,

$$\begin{aligned}\int_0^s \int_0^{\infty} &\mathbb{P}(T_{2n} \in dv) \mathbb{P}\left(X > x - (K+\epsilon)(u+v)\right) du \\ &\geq \int_0^s \int_0^{\infty} \mathbb{P}(T_{2n} \in dv)(1-\delta)\mathbb{P}(X > x - (K+\epsilon)v) du.\end{aligned}$$

Then, evaluating the integrals and noticing the integrand on the right hand side is constant with respect to u gives, for $x > R(s)$,

$$\int_0^s \int_0^{\infty} \mathbb{P}(T_{2n} \in dv) \mathbb{P}\left(X > x - (K+\epsilon)(u+v)\right) du \geq s(1-\delta)\mathbb{P}(X > x - (K+\epsilon)T_{2n}).$$

Now, consider some $l > 0$ and suppose $x > R(l)$, so that $x > R(s)$ for any $s \in [0, l]$. Then,

$$
\begin{aligned}
\int_0^\infty \mathbb{P}(T_2 \in ds) \int_0^s \int_0^\infty & \mathbb{P}(T_{2n} \in dv)\mathbb{P}(X > x - (K+\epsilon)(u+v))du \\
&\geq \int_0^l \mathbb{P}(T_2 \in ds) s(1-\delta)\mathbb{P}(X > x - (K+\epsilon)T_{2n}) \\
&= (1-\delta)\mathbb{P}(X > x - (K+\epsilon)T_{2n})\mathbb{E}[T_2; T_2 < l].
\end{aligned}
$$

Since $l, \delta > 0$ are arbitrary and T_2 is integrable, l can be taken sufficiently large to obtain $\mathbb{E}[T_2; T_2 < l] \geq (1-\delta)\mathbb{E}[T_2]$, hence,

$$
\begin{aligned}
\int_0^\infty \mathbb{P}(T_2 \in ds) \int_0^s \int_0^\infty & \mathbb{P}(T_{2n} \in dv)\mathbb{P}(X > x - (K+\epsilon)(u+v))du \\
&\geq (1-\delta)^2\mathbb{P}(X > x - (K+\epsilon)T_{2n})\mathbb{E}[T_2].
\end{aligned}
$$

Substituting this into the expression for G_X, for $x > R(l)$, yields

$$
G_X(x) \geq (1-\delta)^2 |K+\epsilon| \mathbb{E}[T_2] \sum_{n=0}^\infty \mathbb{P}(X > x - (K+\epsilon)T_{2n}).
$$

For the upper bound, since $-(K+\epsilon)u > 0$ for $u > 0$,

$$
\mathbb{E}\left[\int_0^{\tilde{T}_2} \mathbb{P}\left(X > x - (K+\epsilon)(u+T_{2n}) \,\middle|\, \sigma(T_{2n})\right) du\right] \leq \mathbb{E}[\tilde{T}_2]\mathbb{P}\left(X > x - (K+\epsilon)T_{2n}\right),
$$

which substituted into the expression for G_X gives, for all $x > 0$,

$$
G_X(x) \leq \mathbb{E}[T_2]|K+\epsilon| \sum_{n=0}^\infty \mathbb{P}(X > x - (K+\epsilon)T_{2n}).
$$

Combining the upper and lower bounds gives equation (5.3.5).

□

Set $Z_n := \xi_{\sigma_n} - \xi_{\sigma_{n-1}}$ for each $n \geq 1$. We are now in a position to consider the asymptotic behaviour of the tails of Z_n conditioned on M_{n-1} and M_n, under the assumption of Theorem 5.3.1. Recall that $H^{(\alpha)} = H_{\xi_\alpha}(x) + H_{U_{-\alpha}}(x)$, for $\alpha \in \{+,-\}$.

Lemma 5.3.5

Suppose that Y is a Lamperti-Kiu process of strong subexponential type such that $\mathbb{E}[\xi_{T_2}] \in (-\infty, 0)$ and fix $n \in \mathbb{N}$, $\alpha, \beta \in \{+,-\}$. If $\beta \in B$, then

$$
\limsup_{x\to\infty} \frac{\mathbb{P}(Z_n > x \mid M_{n-1} = \alpha, M_n = \beta)}{H^{(\beta)}(x)} \leq \frac{1}{\eta_{\alpha,\beta}|K+\epsilon|\mathbb{E}[T_2]}.
$$

Furthermore, if $\beta \notin B$ and $b \in B$, then

$$\limsup_{x\to\infty} \frac{\mathbb{P}(Z_n > x \mid M_{n-1} = \alpha, M_n = \beta)}{H^{(b)}(x)} = 0.$$

Proof

Suppose that $x > A$, let $u_0 \in (0, x - A)$ and fix $\alpha, \beta \in \{+, -\}$. For ease of notation, let $\sigma := \sigma_1$. Since $\mathbb{P}(Z_n > x \mid M_{n-1} = \alpha, M_n = \beta) = \mathbb{P}_\alpha(\xi_\sigma > x \mid \sigma < \infty,\ J_\sigma = \beta)$, it follows that

$$\mathbb{P}(Z_n > x \mid M_{n-1} = \alpha, M_n = \beta) = \frac{1}{\eta_{\alpha,\beta}} \sum_{m=0}^{\infty} \mathbb{P}_\alpha\left(\xi_\sigma > x;\ T_m \le \sigma < T_{m+1};\ M_1 = \beta\right).$$

To bound the elements of the sum, first consider the strict inequality $T_m < \sigma < T_{m+1}$, for some $m \in \mathbb{N}$. Then,

$$\mathbb{P}_\alpha\left(\xi_\sigma > x;\ T_m < \sigma < T_{m+1}; J_\sigma = \beta\right) \le \mathbb{P}_\alpha\left(\sup_{T_m < u < T_{m+1}} \xi_u > x;\ \xi_{T_m} < (K+\epsilon)T_m + A; J_{T_m} = \beta\right)$$

and using Lemma C.3.2 gives

$$\begin{aligned}\mathbb{P}_\alpha\left(\xi_\sigma > x;\ T_m < \sigma < T_{m+1}; J_\sigma = \beta\right) &\le \mathbb{P}_\alpha\left(\sup_{0<u<\tilde{\zeta}_\beta} \tilde{\xi}^{(\beta)}_u > x - (K+\epsilon)T_m - A; J_{T_m} = \beta\right)\\ &\le \frac{\mathbb{P}_\alpha\left(\tilde{\xi}^{(\beta)}_{\tilde{\zeta}_\beta} \ge x - (K+\epsilon)T_m - A - u_0; J_{T_m} = \beta\right)}{\mathbb{P}\left(\xi^{(\beta)}_{\zeta_\beta} \ge -u_0\right)},\end{aligned}$$

where $\tilde{\xi}^{(\beta)}$ and $\tilde{\zeta}_\beta$ are independent identically distributed copies of the Lévy process $\xi^{(\beta)}$ and the exponential random variable ζ_β, respectively.

In the case that $\sigma = T_m$,

$$\begin{aligned}\mathbb{P}_\alpha\left(\xi_\sigma > x;\ T_m = \sigma; J_\sigma = \beta\right) &\le \mathbb{P}_\alpha\left(\xi_{T_m} > x;\ \xi_{T_m-} \le (K+\epsilon)T_m + A;\ J_{T_m} = \beta\right)\\ &\le \mathbb{P}_\alpha\left(\xi_{T_m} - \xi_{T_m-} > x - (K+\epsilon)T_m - A; J_{T_m} = \beta\right)\\ &= \mathbb{P}_\alpha\left(U_{-\beta} > x - (K+\epsilon)T_m - A; J_{T_m} = \beta\right).\end{aligned}$$

If $\alpha = \beta$, then there must be an even number of changes of J before σ, so there exists $m \in \mathbb{N}$ such that $\sigma \in [T_{2m}, T_{2m+1})$. Hence combining the two results above gives,

$$\begin{aligned}\mathbb{P}\left(Z_n > x \mid M_{n-1} = \alpha, M_n = \beta\right) \le{}& \frac{1}{\eta_{\alpha,\beta}} \sum_{m=0}^{\infty} \frac{\mathbb{P}_\alpha\left(\tilde{\xi}^{(\beta)}_{\tilde{\zeta}_\beta} \ge x - (K+\epsilon)T_{2m} - A - u_0\right)}{\mathbb{P}\left(\xi^{(\beta)}_{\zeta_\beta} > -u_0\right)}\\ &+ \frac{1}{\eta_{\alpha,\beta}} \sum_{m=0}^{\infty} \mathbb{P}_\alpha\left(U_{(-\beta)} \ge x - (K+\epsilon)T_{2m} - A - u_0\right).\end{aligned}$$

If $\alpha \neq \beta$, then J changes an odd number of times before time σ. However, $T_{2m+1} \geq T_{2m}$ so the inequalities can be weakened to give the same result as the $\alpha = \beta$ case.

For ease of notation define

$$W_\beta(u_0) := \mathbb{P}\left(\xi^{(\beta)}_{\zeta_\beta} \geq -u_0\right) \leq 1.$$

If $\xi^{(\beta)}_{\zeta_\beta}$ is long tailed, then Lemma 5.3.4 can be used to obtain the asymptotic approximation

$$\sum_{m=0}^{\infty} \mathbb{P}_\alpha\left(\tilde{\xi}^{(\beta)}_{\tilde{\zeta}_\beta} \geq x - (K+\epsilon)T_{2m} - A - u_0\right) \sim \frac{G_{\xi_\beta}(x)}{|K+\epsilon|\mathbb{E}[T_2]},$$

as $x \to \infty$. Similarly, if $U_{-\beta}$ is long tailed, then

$$\sum_{m=0}^{\infty} \mathbb{P}_\alpha\left(U_{-\beta} > x - (K+\epsilon)T_{2m} - A - u_0\right) \sim \frac{G_{U_{-\beta}}(x)}{|K+\epsilon|\mathbb{E}[T_2]},$$

as $x \to \infty$. The cases where both of the identities hold, exactly one holds or neither hold are considered separately below.

In the case where both $\xi^{(\beta)}_{\zeta_\beta}$ and $U_{-\beta}$ are strong subexponential (and hence are long-tailed) for all $\delta > 0$, there exists an $R > 0$ such that for all $x > R$,

$$\begin{aligned}
&\frac{\mathbb{P}\left(Z_n > x | M_{n-1} = \alpha, M_n = \beta\right)}{G_{\xi_\beta}(x) + G_{U_{-\beta}}(x)} \\
&\qquad \leq \frac{1}{\left(G_{\xi_\beta}(x) + G_{U_{-\beta}}(x)\right)\eta^{(\alpha,\beta)}}\left(\frac{(1+\delta)G_{\xi_\beta}(x)}{W_\beta(u_0)|K+\epsilon|\mathbb{E}[T_2]} + \frac{(1+\delta)G_{U_{-\beta}}(x)}{|K+\epsilon|\mathbb{E}[T_2]}\right) \\
&\qquad \leq \frac{(1+\delta)}{\eta^{(\alpha,\beta)}W_\beta(u_0)|K+\epsilon|\mathbb{E}[T_2]},
\end{aligned}$$

where the second inequality holds since $W_\beta(u_0) < 1$. Since δ was arbitrary, taking the $\limsup$ as $x \to \infty$ yields

$$\limsup_{x\to\infty} \frac{\mathbb{P}\left(Z_n > x \mid M_{n-1} = \alpha, M_n = \beta\right)}{G_{\xi_\beta}(x) + G_{U_{-\beta}}(x)} \leq \frac{1}{\eta_{\alpha,\beta}W_\beta(u_0)|K+\epsilon|\mathbb{E}[T_2]}.$$

In the case where exactly one of $\xi^{(\beta)}_{\zeta_\beta}$ and $U_{-\beta}$ is strong subexponential, it asymptotically dominates the other as $x \to \infty$. Suppose that it is $\xi^{(\beta)}_{\zeta_\beta}$ that is subexponential and note that the following argument is symmetric in $\xi^{(\beta)}_{\zeta_\beta}$ and $U_{-\beta}$. For all $\delta > 0$, there exists $\hat{\delta} > 0$ such that $\hat{\delta}(1+\hat{\delta}) < \delta/2$. Also, there exists an $R > 0$, such that for all $x > R$ and $n \in \mathbb{N}$,

$$\mathbb{P}_\alpha\left(U_{-\beta} > x - (K+\epsilon)T_{2n} - A - u_0\right) \leq \hat{\delta}\mathbb{P}_\alpha\left(\tilde{\xi}^{(\beta)}_{\tilde{\zeta}_\beta} \geq x - (K+\epsilon)T_{2n} - A - u_0\right).$$

Hence, for all $x > R$, for suitably large R

$$\sum_{m=0}^{\infty} \mathbb{P}_\alpha\left(U_{-\beta} > x - (K+\epsilon)T_{2m} - A - u_0\right) \leq \frac{\hat{\delta}(1+\hat{\delta})G_{\xi_\beta}(x)}{|K+\epsilon|\mathbb{E}[T_2]},$$

which gives for $x > R$,

$$\frac{\mathbb{P}\left(Z_n > x \mid M_{n-1} = \alpha, M_n = \beta\right)}{G_{\xi_\beta}(x) + G_{U_{-\beta}}(x)} \leq \frac{1}{\eta_{\alpha,\beta} G_{\xi_\beta}(x)} \left(\frac{(1+\frac{\delta}{2})G_{\xi_\beta}(x)}{|K+\epsilon|\mathbb{E}[T_2]W_\beta(u_0)} + \frac{\frac{\delta}{2}G_{\xi_\beta}}{|K+\epsilon|\mathbb{E}[T_2]} \right)$$
$$\leq \frac{1+\delta}{\eta_{\alpha,\beta}|K+\epsilon|\mathbb{E}[T_2]W_\beta(u_0)}.$$

Then, since $\delta > 0$ was arbitrary,

$$\limsup_{x\to\infty} \frac{\mathbb{P}\left(Z_n > x \mid M_{n-1} = \alpha, M_n = \beta\right)}{G_{\xi_\beta}(x) + G_{U_{-\beta}}(x)} \leq \frac{1}{\eta_{\alpha,\beta}|K+\epsilon|\mathbb{E}[T_2]W_\beta(u_0)}.$$

Finally, consider the case that neither is strong subexponential. Since the MAP is of strong subexponential type, the tails of $\tilde{\xi}^{(\beta)}_{\tilde{\zeta}_\beta}$ and $U_{-\beta}$ are dominated by the tails of at least one of either $\tilde{\xi}^{(b)}_{\tilde{\zeta}_b}$ or U_{-b}. Denote the dominating random variable by X and let $V \in \left\{\tilde{\xi}^{(\beta)}_{\tilde{\zeta}_\beta}, U_{-\beta}\right\}$. Following the above calculation, for all $\delta > 0$ there exists an $R > 0$ such that for any $x > R$ and $n \in \mathbb{N}$,

$$\mathbb{P}_\alpha\left(V \geq x - (K+\epsilon)T_{2n} - A - u_0\right) \leq \delta \mathbb{P}_\alpha\left(X > x - (K+\epsilon)T_{2n} - A - u_0\right).$$

Then, using the results of the previous two cases, for suitably large $R > 0$,

$$\sum_{n=0}^{\infty} \mathbb{P}_\alpha\left(V \geq x - (K+\epsilon)T_{2n} - A - u_0\right) \leq \frac{\delta(1+\delta)G_X(x)}{|K+\epsilon|\mathbb{E}[T_2]W_b(u_0)}.$$

Hence for $x > R$,

$$\frac{\mathbb{P}\left(Z_n > x \mid M_{n-1} = \alpha, M_n = \beta\right)}{G_{\xi_{-b}}(x) + G_{U_b}(x)} \leq \frac{2\delta(1+\delta)}{\eta_{\alpha,\beta}|K+\epsilon|\mathbb{E}[T_2]W_b(u_0)}$$

and so as $x \to \infty$, since $\delta > 0$ was arbitrary,

$$\mathbb{P}\left(Z_n > x \mid M_{n-1} = \alpha, M_n = \beta\right) = o\left(G_{\xi_b}(x) + G_{U_{-b}}(x)\right).$$

Since all the components of the decomposition (2.3.2) are finite, for sufficiently large $x \in \mathbb{R}^+$, $G_{(\cdot)}(x) = H_{(\cdot)}(x)$. Hence, in the first two cases,

$$\limsup_{x\to\infty} \frac{\mathbb{P}(Z_n > x \mid M_{n-1} = \alpha, M_n = \beta)}{H_{\xi_\beta}(x) + H_{U_{-\beta}}(x)} \leq \frac{1}{\eta_{\alpha,\beta}W(u_0)|K+\epsilon|\mathbb{E}[T_2]}.$$

Since we are taking $x \to \infty$, we may also take $u_0 \to \infty$ and use that $W(u_0) \to 1$ to obtain the result

$$\limsup_{x\to\infty} \frac{\mathbb{P}(Z_n > x \mid M_{n-1} = \alpha, M_n = \beta)}{H_{\xi_\beta}(x) + H_{U_{-\beta}}(x)} \leq \frac{1}{\eta_{\alpha,\beta}|K+\epsilon|\mathbb{E}[T_2]},$$

whilst in the third case

$$\limsup_{x\to\infty} \frac{\mathbb{P}(Z_n > x \mid M_{n-1} = \alpha, M_n = \beta)}{H_{\xi_{-\beta}}(x) + H_{U_\beta}(x)} = 0.$$

$\square$

The following lemma will be needed to consider the asymptotics in Lemma 5.3.7. Since no good reference could be found, a proof is included for completeness.

Lemma 5.3.6

Suppose Z is a real-valued random variable, with tail $\mathbb{P}(Z \geq x)$, and $Q(x)$ is a function, such that $1 - Q(x)$ is a true distribution function. If $\mathbb{P}(Z \geq x) \leq Q(x)$ for all sufficiently large $x \in \mathbb{R}^+$, then there exists a random variable X, which is a function of Z and an independent uniform random variable, such that $Z \leq X$ and $\mathbb{P}(X \geq x) = Q(x)$.

Proof

Let $P(x) := \mathbb{P}(Z \geq x)$ and $V \sim \text{Unif}(0,1)$ be independent of Z. This proof makes use of the notation $P(x^+) := \lim_{y \downarrow x} P(y)$, which exists for all $x \geq 0$ since P is non-increasing and bounded from below. Define the random function $U : \mathbb{R}^+ \to [0,1]$ by setting $U(x) := P(x) - V(P(x) - P(x^+))$. Let $x_1 < x_2$, then since P is non-increasing, $P(x^+) \leq U(x) \leq P(x)$ for all $x \in \mathbb{R}$ and

$$U(x_2) \leq P(x_2) \leq P(x_1^+) = P(x_1) - 1(P(x_1) - P(x_1^+)) \leq U(x_1),$$

hence U is also non-increasing.

Furthermore, suppose $U(x_1) = U(x_2)$ for some $x_1 < x_2$. Then, $P(x_1^+) \leq U(x_1) = U(x_2) \leq P(x_2) \leq P(x_1^+)$, where the last inequality is because P is non-increasing, and so $P(x_1^+) = P(x_2)$. If $P(x_1^+) = P(x_1)$, then $P(x_2) = P(x_1)$. Otherwise,

$$U(x_1) > P(x_1^+) \geq P(x_2) > U(x_2).$$

This is a contradiction. Hence, if $x_1 < x_2$ and $U(x_1) = U(x_2)$, then $P(x_1^+) = P(x_1) = P(x_2)$ a.s. and so $\mathbb{P}(x_1 \leq Z < x_2) = 0$.

From this, it follows that, for all $x \in \mathbb{R}$,

$$\mathbb{P}(U(Z) \leq U(x)) = \mathbb{P}(Z \geq x) + \mathbb{P}(Z < x\,;\, U(Z) = U(x)) = P(x) + \mathbb{P}(Z < x\,;\, U(Z) = U(x)).$$

However, by the above calculation,

$$\mathbb{P}(Z < x\,;\, U(Z) = U(x)) \leq \mathbb{P}(Z < x\,;\, P(Z) = P(x)) = 0$$

and hence, for all $x \in \mathbb{R}$, we have $\mathbb{P}(U(z) \leq U(x)) = P(x)$.

Now, let $q \in [0,1]$ and suppose that there exists $x \in \mathbb{R}^+$ with $P(x^+) = P(x) = q$ so

$$\mathbb{P}(U(Z) \leq q) = \mathbb{P}(U(Z) \leq P(x)) = \mathbb{P}(U(Z) \leq U(x)) = P(x) = q.$$

If there is not such an x, then since $\lim_{x\to-\infty} P(x) = 1$ and $\lim_{x\to\infty} P(x) = 0$, there exists $x \in \mathbb{R}^+$ such that $q \in [P(x^+), P(x))$. If $y > x$, then $U(y) \leq P(y) \leq P(x^+)$, and so $U(y) \notin (P(x^+), P(x))$. Similarly, if $y < x$, then $U(y) \geq U(y^+) \geq P(x)$, so $U(y) \notin (P(x^+), P(x))$. Hence, $U(Z) \in (q, P(x)) \subset (P(x^+), P(x))$ implies $Z = x$.

Next consider $\mathbb{P}(P(x) = U(Z)\,;\, Z \neq x)$. Notice that if $z > x$, then $U(z) \leq P(x^+) < P(x)$ and so $\mathbb{P}(P(x) = U(z)\,;\, Z > x) = 0$. If $z < x$ and $P(x) = U(z)$, then $P(x) = U(z) \geq P(z^+) \geq P(x)$ and so $P(x) = P(z^+)$. However, if there is a discontinuity point $y \in [z, x)$, then $U(z) \geq P(z^+) \geq P(y) > P(y^+) \geq P(x)$ so $\mathbb{P}(P(x) = U(z)\,;\, Z < x) = 0$ hence $P(z) = P(z^+) = P(x)$ and $\mathbb{P}(P(x) = U(Z)\,;\, Z \neq x) = 0$.

From this and since V is uniformly distributed on $[0,1]$,

$$\begin{aligned}
\mathbb{P}(U(Z) \in (q, P(x)]) &= \mathbb{P}(Z = x)\mathbb{P}(U(x) \in (q, P(x)]) + \mathbb{P}(Z \neq x; U(Z) = P(x)) \\
&= \mathbb{P}(Z = x)\mathbb{P}\left(\frac{q - P(x^+)}{P(x) - P(x^+)} < V \leq \frac{P(x) - P(x^+)}{P(x) - P(x^+)}\right) + 0 \\
&= \left(P(x) - P(x^+)\right)\frac{q - P(x^+)}{P(x) - P(x^+)},
\end{aligned}$$

hence,

$$\begin{aligned}
\mathbb{P}(U(Z) \leq q) &= \mathbb{P}(U(Z) \leq P(x)) - \mathbb{P}(U(Z) \in (q, P(x)]) \\
&= P(x) - (P(x) - P(x^+))\frac{(P(x) - q)}{P(x) - P(x^+)} = q.
\end{aligned}$$

Hence, U is uniformly distributed on $[0,1]$. Using [21, pp 7, Proposition 3.1], the random variable defined by $X := U^{-1}(Z) := \inf\{x \in \mathbb{R}^+ \mid Q(x) < U(Z)\}$ has the distribution Q. Moreover, since $Q(x) \geq P(x)$ for all $x \in \mathbb{R}^+$,

$$Q(Z) \geq P(Z) \geq P(Z) - V(P(Z) - P(Z^+)) = U(Z)$$

and, since Q is non-increasing, $X = \inf\{x \in \mathbb{R}^+ \mid Q(x) < U(Z)\} \geq Z$, as required. $\square$

The following lemma provides the upper bound of Theorem 5.3.1.

Lemma 5.3.7

If (J, ξ) is a MAP of strong subexponential type and $\mathbb{E}[\xi_{T_2}] \in (-\infty, 0)$, then the right tail of A_∞ satisfies

$$\limsup_{x\to\infty} \frac{\mathbb{P}(\log(A_\infty) > x)}{H(x)} \leq \frac{1}{|\mathbb{E}[\xi_{T_2}]|}, \tag{5.3.7}$$

where H is the function from Theorem 5.3.1.

Proof

Fix $\sigma \in \{+,-\}$ and let $\delta_2 > 0$. For sufficiently large $A > 0$, by Proposition 5.3.1, $\mathbb{E}_\sigma[N(\alpha,\beta)]/\eta_{\alpha,\beta} \leq \mathbb{1}_{\{\sigma=\alpha\}} + \delta_2$. Now fix such an $A > 0$ and let $\delta_1 > 0$.

From Lemma 5.3.2, $\log(A_\infty) \leq (N+1)C + \sum_{i=1}^N Z_i^+$. For each $i \in \mathbb{N}$, applying Lemma 5.3.6 to the tail estimate of Z_i^+ given $\{M_n\}_{n\in\mathbb{N}_0}$ from Lemma 5.3.5 yields a random variable $X_i(k)$, for each $k \in \cup_{n\in\mathbb{N}}\{+,-\}^n$ with $k_0 = \sigma$, such that:

1. each $X_i(k)$ is a function of Z_i and a random variable independent of the rest of the system;
2. $X_i(k) \geq (Z_i^+ + C)\mathbb{1}_{\{N=n;(M_0,\cdots,M_n)=k\}}$;
3. $X_i(k)$ has tails given by $\min\left(1, H^{(k_i)}(x)(\eta_{k_{i-1},k_i}|K + \epsilon|\mathbb{E}[T_2]|)^{-1}\right)$, if $k_i \in B$, and tails which are $o(\min(1, H^{(b)}(x)))$ for $b \in B$, if $k_i \notin B$.

Then, summing up over the sample paths of $\{M_n\}_{n\in\mathbb{N}_0}$ gives the upper bound

$$\mathbb{P}_\sigma\left(\sum_{i=1}^N Z_i^+ + C > x\right) \leq \sum_{n\in\mathbb{N}} \sum_{\substack{k\in\{+,-\}^{n+1}\\ k_0=\sigma}} \mathbb{P}_\sigma\left(\sum_{i=1}^n X_i(k) > x; N = n; (M_0,\cdots,M_n) = k\right).$$

For ease of notation, let $\bar{\eta} = \max_{\alpha,\beta\in\{+,-\}} \eta_{\alpha,\beta}$, $\underline{\eta} = \min_{\alpha,\beta\in\{+,-\}} \eta_{\alpha,\beta}$ and $d \in (0, (1 - 2\bar{\eta})/2\bar{\eta})$.

For each $\alpha, \beta \in \{+,-\}$, $n \in \mathbb{N}$ and $k \in \{+,-\}^{n+1}$, such that $k_0 = \sigma$, let $n_{\alpha,\beta}(k) := \sum_{i=0}^n \mathbb{1}_{\{k_{i-1}=\alpha,k_i=\beta\}}$. Then, by Lemma 5.3.3, given the event $\{N = n; (M_0,\ldots,M_n) = k\}$, the sum $Y_{\alpha,\beta}(k) := \sum_{i=1}^n X_i \mathbb{1}_{\{k_{i-1}=\alpha,k_i=\beta\}}$ is a sum of $n_{\alpha,\beta}(k)$ i.i.d. random variables. Hence, in the case $\beta \in B$, from Kesten's bound [24, pp 67, Section 3.10, Theorem 3.34], there exists a constant $c(d) > 0$, such that

$$\begin{aligned}&\mathbb{P}_\sigma\left(Y_{\alpha,\beta}(k) > x \mid N = n;\ (M_0,\ldots,M_n) = k\right)\\ &\qquad\leq c(d)(1+d)^{n_{\alpha,\beta}(k)}\mathbb{P}_\sigma(X_1((\alpha,\beta)) > x) \leq \frac{c(d)(1+d)^n H(x)}{\underline{\eta}|K + \epsilon|\mathbb{E}[T_2]}.\end{aligned}$$

In the case $\beta \notin B$, since the MAP is of strong subexponential type, for any $b \in B$,

$$\begin{aligned}&\mathbb{P}_\sigma\left(Y_{\alpha,\beta}(k) > x \mid N = n;\ (M_0,\ldots,M_n) = k\right)\\ &\qquad\leq \mathbb{P}_\sigma\left(\sum_{i=1}^n W_i \mathbb{1}_{\{k_{i-1}=\alpha,k_i=\beta\}} \,\middle|\, N = n;\ (M_0,\ldots,M_n) = k\right) \leq \frac{c(d)(1+d)^n H(x)}{\underline{\eta}|K + \epsilon|\mathbb{E}[T_2]},\end{aligned}$$

where, for each $i \in \mathbb{N}$, the random variable W_i depends only on $X_i(k)$ and has the distribution of $X_1((\alpha, b))$. Now, using corollaries 3.16 and 3.18 in [24, pp 52, Chapter 3] to sum the $Y_{\alpha,\beta}$ gives the bound

$$\mathbb{P}_\sigma\left(\sum_{i=1}^{n} X_i(k) > x \,\middle|\, N = n; (M_0, \cdots, M_n) = k\right)$$
$$= \mathbb{P}_\sigma\left(\sum_{\alpha,\beta\in\{+,-\}} Y_{\alpha,\beta}(k) > x \,\middle|\, N = n; (M_0, \cdots, M_n) = k\right) \leq \frac{4c(d)(1+d)^n H(x)}{\underline{\eta}|K + \epsilon|\mathbb{E}[T_2]}.$$

Using the bound on the distribution of N from (5.3.3), there is an $M \in \mathbb{N}$ such that $\mathbb{E}[\underline{\eta}^{-1}4c(d)(1+d)^N; N > M] \leq \delta_1$, for sufficiently small d. It follows that,

$$\mathbb{P}_\sigma\left(\sum_{i=1}^{N} X_i(\{M_n\}_{n\in\mathbb{N}}) > x; N > M\right) \leq \frac{\delta_1 H(x)}{|K + \epsilon|\mathbb{E}[T_2]}.$$

Moreover, by [24, pp 52, Chapter 3, Corollary 3.16], for all $n \leq M$ and $k \in \{+,-\}^{n+1}$ with $k_0 = \sigma$, there exists $R_{n,k} > 0$ such that, for all $x > R_{n,k}$,

$$\mathbb{P}_\sigma\left(\sum_{i=1}^{n} X_i(k) > x \,\middle|\, N = n; (M_0, \cdots, M_n) = k\right)$$
$$\leq \left(\left(1 + \frac{\delta_1}{2}\right)\sum_{\alpha\in\{+,-\}}\sum_{\beta\in B} n_{\alpha,\beta}(k)\frac{H^\beta(x)}{\eta_{\alpha,\beta}|K + \epsilon|\mathbb{E}[T_2]}\right) \bar{*} \mathbb{P}_\sigma\left(\sum_{i=1}^{n} \mathbb{1}_{\{k_i\notin B\}} X_i(k) > \cdot\right)(x),$$

where, for two survival functions $\bar{U}$ and $\bar{V}$, $\bar{U}\bar{*}\bar{V}$ denotes the survival function of the convolution of the distribution functions $V := 1 - \bar{V}$ and $U := 1 - \bar{U}$. Then, since $\mathbb{P}_\sigma\left(\mathbb{1}_{\{k_i\notin B\}} X_i(k)) > x\right) = o\left(H^\beta(x)\right)$ for any $\beta \in B$, by [24, pp 52, Chapter 3, Corollary 3.18],

$$\mathbb{P}_\sigma\left(\sum_{i=1}^{n} X_i(k) > x \,\middle|\, N = n; (M_0, \cdots, M_n) = k\right)$$
$$\leq (1 + \delta_1)\sum_{\alpha\in\{+,-\}}\sum_{\beta\in B} n_{\alpha,\beta}(k)\frac{H^\beta(x)}{\eta_{\alpha,\beta}|K + \epsilon|\mathbb{E}[T_2]}.$$

Since there are finitely many such pairs (n, k) with $n \leq M$, take
$R := \max_{n\leq M} \max_{k\in\{+,-\}^{n+1}} R_{n,k}$. Then, for all $x > R$,

$$\sum_{n=1}^{M} \sum_{\substack{k\in\{+,-\}^{n+1}\\ k_0=\sigma}} \mathbb{P}_\sigma\left(\sum_{i=1}^{n} X_i(k) > x; N=n; (M_0,\cdots,M_n)=k\right)$$
$$\leq (1+\delta_1)\sum_{n=1}^{M} \sum_{\substack{k\in\{+,-\}^{n+1}\\ k_0=\sigma}} \sum_{\alpha\in\{+,-\}} \sum_{\beta\in B} n_{\alpha,\beta}(k) \frac{H^{(\beta)}(x)}{\eta_{\alpha,\beta}|K+\epsilon|\mathbb{E}[T_2]} \mathbb{P}_\sigma(N=n; (M_0,\cdots,M_n)=k)$$
$$= (1+\delta_1) \sum_{\alpha\in\{+,-\}} \sum_{\beta\in B} \mathbb{E}_\sigma[n_{\alpha,\beta}(k); N\leq M] \frac{H^{(\beta)}(x)}{\eta\alpha,\beta|K+\epsilon|\mathbb{E}[T_2]}$$
$$\leq \frac{(1+\delta_1)}{|K+\epsilon|\mathbb{E}[T_2]} \sum_{\alpha\in\{+,-\}} \sum_{\beta\in B} \frac{\mathbb{E}_\sigma[n_{\alpha,\beta}(k)]H^{(\beta)}(x)}{\eta\alpha,\beta}.$$

Since A was chosen such that $\mathbb{E}_\sigma[n(\alpha,\beta)]/\eta_{\alpha,\beta} \leq \mathbb{1}_{\sigma=\alpha} + \delta_2$, this leads to

$$\sum_{n=1}^{M} \sum_{\substack{k\in\{+,-\}^{n+1}\\ k_0=\sigma}} \mathbb{P}_\sigma\left(\sum_{i=1}^{n} X_i(k) > x; N=n; (M_0,\cdots,M_n)=k\right)$$
$$\leq \frac{(1+\delta_1)}{|K+\epsilon|\mathbb{E}[T_2]} \sum_{\alpha\in\{+,-\}} \sum_{\beta\in B} \left(\mathbb{1}_{\{\sigma=\alpha\}} + \delta_2\right) H^{(\beta)}(x).$$

Computing the sum then gives

$$\sum_{n=1}^{M} \sum_{\substack{k\in\{+,-\}^{n+1}\\ k_0=\sigma}} \mathbb{P}_\sigma\left(\sum_{i=1}^{n} X_i(k) > x; N=n; (M_0,\cdots,M_n)=k\right)$$
$$= \frac{(1+\delta_1)}{|K+\epsilon|\mathbb{E}[T_2]}(1+2\delta_2)\sum_{\beta\in B} H^{(\beta)}(x) \leq \frac{(1+\delta_1)^2}{|K+\epsilon|\mathbb{E}[T_2]}(1+2\delta_2)H(x),$$

where the last inequality holds for sufficiently large x since the MAP is of strong subexponential type. Hence, for all $x > R$,

$$\mathbb{P}_\sigma\left(\sum_{i=1}^{N}\left(Z_i^+ + C\right) > x\right) \leq \frac{(1+\delta_1)^2(1+2\delta_2)}{|K+\epsilon|\mathbb{E}[T_2]}H(x) + \frac{\delta_1 H(x)}{|K+\epsilon|\mathbb{E}[T_2]},$$

and so from the definition of lim sup,

$$\limsup_{x\to\infty} \frac{\mathbb{P}_\sigma\left(C + \sum_{i=1}^{N}\left(Z_i^+ + C\right) > x\right)}{H(x-C)} \leq \frac{1+2\delta_2}{|K+\epsilon|\mathbb{E}[T_2]}.$$

However, since H is long tailed, $\lim_{x\to\infty} H(x-C)/H(x) = 1$ and therefore,

$$\limsup_{x\to\infty} \frac{\mathbb{P}_\sigma\left(C + \sum_{i=1}^{N}\left(Z_i^+ + C\right) > x\right)}{H(x)} \leq \frac{1+2\delta_2}{|K+\epsilon|\mathbb{E}[T_2]}.$$

Then, by comparison

$$\limsup_{x\to\infty} \frac{\mathbb{P}_\sigma(\log(A_\infty) > x)}{H(x)} \leq \frac{1+2\delta_2}{|K+\epsilon|\mathbb{E}[T_2]},$$

and since both ϵ and δ_2 were arbitrary, the result follows. □

It remains to show the lower bound for $\liminf_{x\to\infty} \mathbb{P}(A_\infty > x)$ also holds. To this end, for each $x \in \mathbb{R}$, define the stopping time

$$\tau_d(x) := \inf\{T_{2n} \mid n \in \mathbb{N}, \xi_{T_{2n}} \geq x\} \tag{5.3.8}$$

and notice that $\tau_d(x) < \infty$ if and only if $\sup_{n\in\mathbb{N}} \xi_{T_{2n}} > x$. Furthermore, $J_{\tau_d(x)} = J_0$ whenever $\tau_d(x) < \infty$. Since ξ_{T_2} is strong subexponential, its integrated tail, H, is a subexponential function and thus, by [55, pp 2, Theorem 1(ii)], $\mathbb{P}(\tau_d(x) < \infty)$ is also a subexponential function. Then, considering the random walk $\{\xi_{T_{2n}}\}_{n\in\mathbb{N}}$ in the place of $\{\xi_n\}_{n\in\mathbb{N}}$ in the proof of [39, pp 12, Lemma 4.3], we have for every $y > 0$,

$$\lim_{x\to\infty} \mathbb{P}\left(\xi_{\tau_d(x)} - x > y \,\middle|\, \tau_d(x) < \infty\right) = 1. \tag{5.3.9}$$

We are now able to prove Theorem 5.3.1.

Proof of Theorem 5.3.1

Equation (5.3.4) of Theorem 5.3.1 follows from the inequality (5.3.7) of Lemma 5.3.7 and the inequality

$$\liminf_{x\to\infty} \frac{\mathbb{P}(\log(A_\infty) > x)}{H(x)} \geq \frac{1}{|\mathbb{E}[\xi_{T_2}]|} = \frac{1}{\mathbb{E}[T_2]K}, \tag{5.3.10}$$

which is proven below.

The following inequality is immediate:

$$\mathbb{P}(\log A_\infty > x) \geq \mathbb{P}\left(\log\left(\int_{\tau_d(x)}^{\infty} |Y_t| dt\right) > x; \tau_d(x) < \infty\right).$$

Then, applying the Markov additive property and recalling $J_{\tau_d(x)} = J_0$ gives

$$\mathbb{P}(\log A_\infty > x) \geq \mathbb{P}\left(\xi_{\tau_d(x)} + \log(\hat{A}_{\infty,J_0}) > x \,\middle|\, \tau_d(x) < \infty\right) \mathbb{P}(\tau_d(x) < \infty),$$

where $\hat{A}_{\infty,\alpha}$ is an independent and identically distributed copy of A_∞ with $\hat{J}_0 = \alpha$. Then, by applying equation (5.3.9),

$$\liminf_{x\to\infty} \frac{\mathbb{P}(\log A_\infty > x)}{H(x)} \geq \liminf_{x\to\infty} \frac{\mathbb{P}(\tau_d(x) < \infty)}{H(x)} = \liminf_{x\to\infty} \frac{\mathbb{P}(\sup_{n\in\mathbb{N}} \xi(T_{2n}) \geq x)}{H(x)}.$$

However, $\xi_{T_{2n}}$ is a sum of the random variables $\xi_{T_{2m}} - \xi_{T_{2(m-1)}}$ which are i.i.d. copies of ξ_{T_2}. Since (J, ξ) is strong subexponential type, the integrated tail, H, of ξ_{T_2} is long-tailed. Then, from Veraverbeke's theorem [55, pp 2, Theorem 1(i)] it follows that

$$\liminf_{x\to\infty} \frac{\mathbb{P}\left(\sup_{n\in\mathbb{N}} \xi_{T_{2n}}\right) \geq x)}{H(x)} \geq \frac{1}{|\mathbb{E}[\xi_{T_2}]|} = \frac{1}{|K|\mathbb{E}[T_2]}.$$

$\square$

Chapter 6

Applications to Financial Modelling

Within this chapter, MAPs and their exponential functionals are applied to the valuation of *European* and *Asian* options. In particular, a model is considered where an asset has a price process, $\{Y_t : t \geq 0\}$, such that for some Markov chain J the process $(J, \log(Y))$ is a MAP. This is referred to as an *exponential MAP model* and the Markov chain J is interpreted as the 'state' of the market. Within the pricing of Asian options, the results of the previous chapters regarding the exponential functional of a MAP will be useful.

It is shown in [42] that in an exponential MAP model, there exists a suitably enlarged market for which there is an *equivalent martingale measure*. Let $\mathbb{P}$ denote this measure and $\{\mathcal{F}_t\}_{t\geq 0}$ be the filtration of the enlarged market. For consistency with the preceding chapters, let $\xi_t := \log(Y_t)$ for all $t \geq 0$. Throughout it is assumed that the *risk free rate* of interest is fixed at r.

6.1 European Option Pricing

A *European option* on an asset with price process $\{Y_t : t \geq 0\}$ is a contract which at its *maturity*, some fixed time $T \geq 0$, pays out $H(Y_T)$, where the *payoff function*, $H : \mathbb{R}^+ \to \mathbb{R}$, is predetermined. In the case of a *European call option*, the owner of the option has the right, but not the obligation, to buy the asset at some predetermined *strike price* k at maturity. Hence, the payoff function for a call option is $\max(x - k, 0)$.

By standard no arbitrage arguments, the price of a European option, with payoff H and maturity T, at time $t \in (0, T)$ is given by

$$e^{-r(T-t)}\mathbb{E}\left[H(Y_T) \mid \mathcal{F}_t\right].$$

From the Markov additive property, this is a function of the current value of the MAP, (J_t, ξ_t), the time to maturity, $T - t$, and is given by

$$e^{-r(T-t)}\mathbb{E}_{(J_t,\xi_t)}\left[H\left(\hat{Y}_{T-t}\right)\right],$$

where $(\hat{J}, \hat{Y})$ is an independent and identically distributed copy of (J, Y). Let $C_H(y, \alpha, \tau)$ denote the price of the European option, with payoff function H and time until maturity τ, when the current market state is given by $(\alpha, y) \in E \times \mathbb{R}^+$. That is, for $(\alpha, y) \in E \times \mathbb{R}^+$ and $0 \leq \tau$,

$$C_H(\alpha, y, \tau) = e^{-r\tau}\mathbb{E}_{(\alpha,\log(y))}\left[H\left(\hat{Y}_\tau\right)\right].$$

Under an exponential Lévy model, two common techniques for pricing European options are integral transform methods, for example see [10] and [23], and solving a Partial Integro-Differential Equation (PIDE), for example see [52, Chapter 12]. These two methods are adapted to exponential MAP models in the following sections.

6.1.1 Mellin Transform Approach

Similarly to the *Fourier transform* methods used for Lévy process (for example, see [10] and [23]), a *Mellin transform* approach can be used to price European options under an exponential MAP model.

First consider the case of *call* and *put* options, which have payoff functions $H_+(x) \coloneqq (x-k)^+$ and $H_-(x) \coloneqq (k-x)^+$, respectively. Let $C_\alpha(k, y) \coloneqq C_{H_+}(\alpha, y, T)$ and $P_\alpha(k, y) \coloneqq C_{H_-}(\alpha, y, T)$ denote the prices of European call and put options, respectively, with strike $k > 0$ and time to maturity $T > 0$, when the current price of the asset is $y \in \mathbb{R}^+$ and the market state is given by $\alpha \in E$

Proposition 6.1.1 (Mellin Transform Pricing of European Call and Put Options)
If $\mathbb{E}[Y_T^{1+s}] < \infty$ for some $s > 0$, then, for all $\alpha \in E$ and $u \in \mathbb{C}$ with $\Re(u) \in (0, s)$,

$$\int_0^\infty k^{u-1}C_\alpha(k, y)\,dk = \frac{e^{-rT}y^{u+1}}{u(u+1)}\sum_{\beta\in E}\left(e^{TF(u+1)}\right)_{\alpha,\beta}, \tag{6.1.1}$$

and, if $\mathbb{E}[Y_T^{-s}] < \infty$ *for some* $s > 0$*, then, for all* $\alpha \in E$ *and* $u \in \mathbb{C}$ *with* $\Re(u) \in (-s, -1)$*,*

$$\int_0^\infty k^{u-1} P_\alpha(k, y)\, dk = \frac{e^{-rT} y^{u+1}}{u(u+1)} \sum_{\beta \in E} \left(e^{TF(u+1)} \right)_{\alpha,\beta}. \qquad (6.1.2)$$

Remark 6.1.1

Notice that each of (6.1.1) and (6.1.2) has the same right hand side, but the region in which these equations are valid is different.

Proof

For each $\alpha, \beta \in E$, let $p_{\alpha,\beta}$ denote the density of $Y_T \mathbb{1}_{\{J_T = \beta\}}$ under the measure $\mathbb{P}_{\alpha,0}$. Then, for $u \in \mathbb{C}$ with $\Re(u) \in (0, s)$,

$$e^{rT} \int_0^\infty k^{u-1} C_\alpha(k, y) dk = \int_0^\infty k^{u-1} \sum_{\beta \in E} \mathbb{E}_{\alpha, \log(y)} \left[(Y_T - k)^+ ; J_T = \beta \right] dk$$

$$= \sum_{\beta \in E} \int_0^\infty k^{u-1} \int_0^\infty (yx - k)^+ p_{\alpha,\beta}(x)\, dx\, dk = \sum_{\beta \in E} \int_0^\infty k^{u-1} \int_{k/y}^\infty y(x - k/y) p_{\alpha,\beta}(x)\, dx\, dk.$$

Since $\Re(u) \in (0, s)$, using Fubini's theorem yields,

$$\int_0^\infty k^{u-1} \int_{k/y}^\infty y(x - k/y) p_{\alpha,\beta}(x)\, dx\, dk$$
$$= \int_0^\infty \int_0^{xy} k^{u-1} y(x - k/y) p_{\alpha,\beta}(x)\, dk\, dx = \int_0^\infty \frac{(xy)^{u+1} p_{\alpha,\beta}(x)}{u(u+1)} dx.$$

Then, from the definition of $e^{TF(u+1)}$,

$$e^{rT} \int_0^\infty k^{u-1} C_\alpha(k, y) dk = \sum_{\beta \in E} \int_0^\infty \frac{(xy)^{u+1} p_{\alpha,\beta}(x)}{u(u+1)} dx = \sum_{\beta \in E} \frac{y^{u+1} \left(e^{TF(u+1)} \right)_{\alpha,\beta}}{u(u+1)}.$$

A similar calculation, with $\Re(u) \in (-s, -1)$, yields the result for the put option.

□

By taking the Mellin transform with respect to the current asset price, a European option with a general payoff function $H : \mathbb{R}^+ \to \mathbb{R}$ can be considered.

Proposition 6.1.2 (Mellin Transform Pricing of European Options)

Suppose that the price process of an asset is given by $\{Y_t : t \geq 0\}$ *with initial condition* $(J_0, Y_0) = (\alpha, y) \in E \times \mathbb{R}^+$*. Let* $H : \mathbb{R}^+ \to \mathbb{R}$ *be the payoff function of a European option with maturity* $T > 0$ *and suppose that* $s \in \mathbb{R}$ *such that* $\mathbb{E}[Y_T^{-s}] < \infty$ *and* $\{\mathcal{M}H\}(s)$ *exists. Then, the Mellin transform of* $C_H(\alpha, y, T)$*, with respect to* y*, is given by*

$$\{\mathcal{M}C_H(\alpha, \cdot, T)\}(z) = e^{-rT} \{\mathcal{M}H\}(z) \sum_{\beta \in E} \left(e^{TF(-z)} \right)_{\alpha,\beta},$$

for all $\alpha \in E$ and $z \in \mathbb{C}$ such that $\Re(z) = s$.

Proof

Fix $\alpha \in E$. Considering the Mellin transform of $C_H(\alpha, y, T)$ with respect to y and using the Markov additive property, for $z \in \mathbb{C}$ with $\Re(z) \in (s - \epsilon, s + \epsilon)$,

$$e^{rT} \left\{\mathcal{M}C_H\right\}(z) = e^{rT} \int_0^\infty x^{z-1} C_H(\alpha, x, T)\, dx = \int_0^\infty x^{z-1} \sum_{\beta \in E} \int_0^\infty H(xu) p_{\alpha,\beta}(u)\, du\, dx,$$

where $p_{\alpha,\beta}$ is the density of $\mathbb{1}_{\{J_T = \beta\}} Y_T$ under the measure $\mathbb{P}_\alpha$. Then, using Fubini's theorem and the substitution $y = xu$, yields

$$e^{rT} \left\{\mathcal{M}C_H\right\}(z) = \sum_{\beta \in E} \int_0^\infty p_{\alpha,\beta}(u) u^{-z} \int_0^\infty y^{z-1} H(y) dy du.$$

The integrals can then be separated to give

$$e^{rT} \left\{\mathcal{M}C_H\right\}(z) = \sum_{\beta \in E} \int_0^\infty y^{z-1} H(y) dy \int_0^\infty p_{\alpha,\beta}(u) u^{-z} du = \sum_{\beta \in E} \left\{\mathcal{M}H\right\}(z) \left(e^{TF(-z)}\right)_{\alpha,\beta},$$

where, by the assumptions, both $\{\mathcal{M}H\}(z)$ and $e^{TF(-z)}$ exist for all $z \in \mathbb{C}$ such that $\Re(z) = s$. $\square$

Remark 6.1.2

By standard results for Mellin transforms, the condition that $\left\{\mathcal{M}H\right\}(s)$ exists for some $s \in \mathbb{R}$ is satisfied if there is an $\epsilon > 0$ such that

$$\frac{H(x)}{x^{-s+\epsilon}} \to 0 \quad \text{as} \quad x \to 0^+ \qquad \text{and} \qquad \frac{H(x)}{x^{-s-\epsilon}} \to 0 \quad \text{as} \quad x \to +\infty. \tag{6.1.3}$$

In particular, (6.1.3) is satisfied for any $H : \mathbb{R}^+ \to \mathbb{R}$ that is bounded by a polynomial, including the payoff functions $(x - k)^+$ and $(k - x)^+$ of a call and put, respectively.

6.1.2 PIDE Approach

Another method to obtain the prices of European options under an exponential MAP model is through a *partial integro-differential equation (PIDE)*, similar to the one derived for exponential Lévy models in [52]. To use this method, some regularity results for European option prices are required.

The following proposition gives regularity conditions of $\rho_\alpha^{(t)}$, the density of ξ_t for $t > 0$ when $J_0 = \alpha \in E$, from which the regularity of $p_\alpha^{(t)}$, the density of Y_t when $J_0 = \alpha$, follows.

Proposition 6.1.3

Suppose that for all $\alpha \in E$, either $\sigma_\alpha^2 > 0$ or the Lévy measure, μ_α, satisfies

$$\liminf_{r \downarrow 0} r^{\gamma-2} \int_{[-r,r]} |x^2| \mu_\alpha(dx) > 0, \tag{6.1.4}$$

for some $\gamma \in (0,2)$. Then, $\rho_\beta^{(t)} \in C^\infty(\mathbb{R})$ and $\frac{\partial^n}{\partial x^n}\rho_\beta^{(t)}(x) \in L^1(\mathbb{R})$, for all $n \in \mathbb{N}_0$ and $\beta \in E$.

Proof

From [48, pp 8, Proposition 2.5 (v)], for $\beta \in E$ and $z \in \mathbb{R}$

$$|\exp(\psi_\beta(iz))|^2 = \exp(\psi_\beta(iz) + \psi_\beta(-iz)) = \exp\left(2\int_{\mathbb{R}} (\cos(zx) - 1)\,\mu(dx)\right).$$

Moreover, following [48, pp 190, Chapter 5, Proposition 28.3], under condition (6.1.4), for small enough $r > 0$,

$$\int_{\mathbb{R}} (\cos(zx) - 1)\,\mu_\beta(dx) \le -c_1 z^\gamma,$$

for some constant $c_1 > 0$ and all $z \in \mathbb{R}$. Thus,

$$2\Re(\psi_\beta(iz)) = \log\left(|\exp(\psi_\beta(iz))|^2\right) \le -2c_1 z^\gamma,$$

and so, $|\psi_\beta(iz)| \to \infty$ as $z \to \pm\infty$. In the case that (6.1.4) does not hold, it is assumed that $\sigma_\beta^2 > 0$ and hence $|\psi_\beta(iz)| \to \infty$ as $z \to \pm\infty$ in this case also. However, for each $\alpha, \beta \in E$ and all $z \in \mathbb{R}$, $|G_{\alpha,\beta}(iz)| \le \mathbb{E}\left[\left|U_{\alpha,\beta}^{iz}\right|\right] = 1$. Hence, for all $\epsilon > 0$, there exists $R > 0$ such that for all $|z| > R$ with $z \in \mathbb{R}$, we have $\epsilon\,(F(iz))_{\alpha,\alpha} \ge \sum_{\beta \in E\setminus\{\alpha\}} (F(iz))_{\alpha,\beta}$ and $\epsilon\,(F(iz))_{\alpha,\alpha} \ge \sum_{\beta \in E\setminus\{\alpha\}} (F(iz))_{\beta,\alpha}$. Then, for sufficiently small $\epsilon > 0$, it follows that $\|e^{F(iz)}\|_{L^1} = \max_{\alpha \in E} |e^{\psi_\alpha(iz)}| \le \exp(-2c_1 z^\gamma)$, where $\|\cdot\|$ is the matrix norm induced by the L^1 norm.

Under the assumptions of the proposition, [48, pp 190, Chapter 5, Proposition 28.1] may be applied to obtain that $\rho_\beta^{(t)} \in \mathbb{C}^\infty$ and $\lim_{x\to\pm\infty} \frac{\partial^n}{\partial x^n}\rho_\beta^{(t)}(x) \to 0$ for all $n \in \mathbb{N}_0$ and $\beta \in E$. Moreover, since $\rho_\beta^{(t)} \in C^\infty(\mathbb{R})$, by the fundamental theorem of calculus, for each $n \in \mathbb{N}$,

$$\int_{\mathbb{R}} \frac{\partial^n}{\partial x^n}\rho_\beta^{(t)}(x)dx = \lim_{x\to\infty} \frac{\partial^{n-1}}{\partial x^{n-1}}\rho_\beta^{(t)}(x) - \lim_{x\to-\infty} \frac{\partial^{n-1}}{\partial x^{n-1}}\rho_\beta^{(t)}(x) = 0 - 0,$$

hence $\frac{\partial^n}{\partial x^n}\rho_\beta^{(t)}(x) \in L^1(\mathbb{R})$. $\square$

From this proposition, smoothness of $C_H(\gamma, y, T)$ with respect to y can be deduced.

Corollary 6.1.1

Suppose there exists an $a \in \mathbb{R}$, such that $x^{-(a+1)}H(x) \in L^1(\mathbb{R}^+)$ and $\mathbb{E}[Y_T^{a-1}] < \infty$. Then, *under the conditions of Proposition 6.1.3, the price of the option, $C_H(\alpha, y, T)$, is infinitely continuously differentiable as a function of $y > 0$, for all $T > 0$ and $\alpha \in E$.*

Proof

For all $y \in \mathbb{R}^+$, $T \in \mathbb{R}^+$ and $\alpha \in E$, from the definitions it follows that

$$C_H(\alpha, y, T) = e^{-rT}\mathbb{E}_{\alpha,\log(y)}\left[H(Y_T)\right] = e^{-rT}\int_0^\infty H(x)p_\alpha^{(T)}\left(\frac{x}{y}\right)dx.$$

Then, writing this in convolution form gives

$$\begin{aligned} C_H(\alpha, y, T) &= e^{-rT}y^{a+1}\int_0^\infty \left(x^{-a}H(x)p_\alpha^{(T)}\left(\frac{x}{y}\right)\frac{x^{a+1}}{y^{a+1}}\frac{1}{x}\right)dx \\ &= e^{-rT}y^{a+1}\left(\left\{x^{-a}H(x)\right\} * \left\{p_\alpha^{(T)}(x^{-1})x^{-(a+1)}\right\}\right)(y), \end{aligned}$$

where $*$ denotes the multiplicative convolution. Since $x^{-(a+1)}H(x) \in L^1(\mathbb{R}^+)$, by differentiation of a convolution, C_H is n-times differentiable with respect to y, whenever $p_\alpha^{(T)}(x^{-1})x^{-(a+1)}$ is n-times differentiable with respect to x and $\frac{\partial^m}{\partial x^m}p_\alpha^{(T)}(x^{-1})x^{-(a+1)} \in L^1(\mathbb{R}^+)$, for $m = 1, \ldots, n$. Therefore, the proof of this condition is now given.

By the assumptions of the corollary and a change of variables,

$$\int_0^\infty p_\alpha^{(T)}(x^{-1})x^{-(a+1)}dx = \int_0^\infty p_\alpha^{(T)}(z)z^{a-1}dz = \mathbb{E}\left[Y_T^{a-1}\right] < \infty,$$

hence, $p_\alpha^{(T)}(x^{-1})x^{-(a+1)} \in L^1(\mathbb{R}^+)$. Then, since $p_\alpha^{(T)}$ is sufficiently differentiable by Proposition 6.1.3, repeatedly applying the fundamental theorem of calculus gives $\frac{\partial^n}{\partial x^n}\left(p_\alpha^{(T-t)}(x^{-1})x^{-(a+1)}\right) \in L^1(\mathbb{R}^+)$ for all $n \in \mathbb{N}$.

$\square$

Proposition 6.1.4

Suppose that there exists $c \in \mathbb{R}$, such that $\mathbb{E}\left[Y_T^{-c}\right] < \infty$ and $\{\mathcal{M}H\}(c)$ exists. Moreover, suppose that, for each $\beta \in E$, either $\sigma_\beta^2 > 0$, or the Lévy measure, μ_β, satisfies

$$\liminf_{r\downarrow 0} r^{\epsilon-2}\int_{[-r,r]}|x^2|\mu_\beta(dx) > 0,$$

for some $\epsilon \in (0, 2)$. Then, the time derivative of C_H exists and is continuous for $T > 0$.

Proof

Since $\mathbb{E}\left[Y_T^{-c}\right] < \infty$, for each $\beta \in E$, it follows that $\lim_{x\to\pm\infty} e^{-cx}\rho_\beta^{(T)}(x) = 0$. Moreover, $e^{-cx}\rho_\beta^{(T)}(x) \in C^\infty(\mathbb{R})$ by Proposition 6.1.3, hence the fundamental theorem of calculus yields

$$\int_{\mathbb{R}}\frac{\partial}{\partial x}e^{-cx}\rho_\beta^{(T)}(x)dx = \lim_{x\to\infty}e^{-cx}\rho_\beta^{(T)}(x) - \lim_{x\to-\infty}e^{-cx}\rho_\beta^{(T)}(x) = 0, \tag{6.1.5}$$

and consequently, $\frac{\partial}{\partial x} e^{-cx} \rho_\beta^{(T)}(x) \in L^1(\mathbb{R})$. Then, by induction, $\frac{\partial^n}{\partial x^n} e^{-cx} \rho_\beta^{(T)}(x) \in L^1(\mathbb{R})$, for all $n \in \mathbb{N}$.

By considering the Fourier transform, for each $n \in \mathbb{N}$,

$$\left| u^n \mathbb{E}\left[e^{(-c+iu)\xi_T} \right] \right| = \left| u^n \mathcal{F}\left\{ e^{-cx} \rho_\beta^{(T)}(x) \right\}(-u) \right| = (2\pi)^{-n} \left| \mathcal{F}\left\{ \frac{\partial^n}{\partial x^n} e^{-cx} \rho_\beta^{(T)}(x) \right\}(-u) \right|,$$

and so $u^n \mathbb{E}\left[e^{(-c+iu)\xi_T}\right] \in L^\infty(\mathbb{R})$. Thus, $\mathbb{E}\left[e^{(-c+iu)\xi_T}\right] = o(u^{-n})$ as $u \to \pm\infty$ for all $n \in \mathbb{N}$ and, in particular, $\mathbb{E}[e^{(-c+iu)\xi_T}] \in L^n(\mathbb{R})$ as a function of $u \in \mathbb{R}$, for all $n \in \mathbb{N}$.

For each $\alpha \in E$, differentiating the result of Proposition 6.1.2 with respect to T gives

$$\frac{\partial}{\partial T} \left\{ \mathcal{M}_y C_H(\alpha, y, T) \right\}(s) = e^{-rT} \left\{ \mathcal{M}H \right\}(s) \sum_{\beta \in E} \left((F(-s) - rI)\, e^{TF(-s)} \right)_{\alpha,\beta}, \tag{6.1.6}$$

for all $s \in c + i\mathbb{R}$, since by the assumptions both $\{\mathcal{M}H\}(s)$ and $\mathbb{E}\left[Y_T^{-s}\right]$ exist and are finite.

Suppose $s = c + iu$ with $u \in \mathbb{R}$, then for all $\alpha, \beta \in E$,

$$|G_{\alpha,\beta}(-(c+iu))| = \left| \mathbb{E}\left[e^{-(c+iu)U_{\alpha,\beta}} \right] \right| \le \mathbb{E}\left[\left| e^{-(c+iu)U_{\alpha,\beta}} \right| \right] = \mathbb{E}\left[e^{-cU_{\alpha,\beta}} \right] < \infty,$$

by Theorem 3.1.1, since $\mathbb{E}\left[Y_T^{-c}\right] < \infty$. Hence, $G_{\alpha,\beta}(-s)$ is bounded over $c + i\mathbb{R}$. Since ψ_σ is the (positive) Laplace exponent of a Lévy process, $\psi_\sigma(-(c+iu)) = \mathcal{O}(u^2)$ as $|u| \to \infty$ (see Appendix C.3.1). It was shown above that

$$e^{TF(-(c+iu))} = \left(\mathbb{E}_\alpha \left[e^{-(c+iu)\xi_T}; J_T = \beta \right] \right)_{\alpha,\beta \in E} = o(u^{-n}),$$

for all $n \in \mathbb{N}$ and by assumption $\{\mathcal{M}H\}(-(c+iu))$ is bounded over $-c + i\mathbb{R}$. Combining each of these terms shows that $\frac{\partial}{\partial T} \{\{\mathcal{M}_y C_H(\alpha, y, T)\}(-c+iu)\} \in L^1(\mathbb{R})$ with respect to u and therefore the inverse Mellin transform can be applied along the line $-c + i\mathbb{R}$.

Since $H(x) = \mathcal{O}\left(x^{-c}\right)$ as $x \to 0^+$ and $x \to +\infty$, the Mellin transform $\mathcal{M}H$ is analytic. That $F(s)$ is analytic follows from Proposition 6.1.3. Moreover, it is clear that the result of Proposition 6.1.2 and (6.1.6) are both continuous functions of T. Hence, by the Leibniz integral rule, the derivative can be passed through the integral in the definition of the inverse Mellin transform to obtain

$$\frac{\partial}{\partial T} C_H(\alpha, y, T) = \mathcal{M}_s^{-1} \left\{ e^{-rT} \left\{ \mathcal{M}H \right\}(s) \sum_{\beta \in E} \left((F(-s) - rI)\, e^{TF(-s)} \right)_{\alpha,\beta} \right\}(y).$$

Now suppose $0 < \tau < T$. Since $(e^{F(-(c+iu))})_{\alpha,\beta} \in L^1(\mathbb{R})$ as a function of u, for each $\alpha, \beta \in E$, it follows that there exist real numbers $a < b$ such that $\|e^{F(-(c+iu))}\| < 1$, for all $u \in \mathbb{R} \backslash (a, b)$.

Let $M := \max\left(\sup_{u\in[a,b]} \|e^{F(-(c+iu))}\|, 1\right)$. Then, M is finite, since the interval $[a,b]$ is compact and $\|e^{F(-(c+iu))}\|$ is continuous. Hence, for all $t \in (T-\tau, T+\tau)$,

$$\left\|e^{tF(-(c+iu))}\right\| \leq B(u) := \begin{cases} M^{T+\tau}, & \text{if } u \in [a,b]; \\ \left\|e^{(T-\tau)F(-(c+iu))}\right\|, & \text{otherwise.} \end{cases}$$

Notice that the right-hand side is an L^1 function in u and is constant with respect to $t \in (T-\tau, T+\tau)$. Then, the following bound can be obtained for all $t \in (T-\tau, T+\tau)$:

$$\left|\frac{\partial}{\partial T}\left(\{\mathcal{M}C_H(\alpha,y,T)\}(s)\right)\right| \leq e^{-r(T-\tau)} \left|\{\mathcal{M}H\}(s)\right| \sum_{\beta\in E} \|F(-s) - rI\| B(u) \in L^1(\mathbb{R}).$$

Thus, by using the dominated convergence theorem in the integral of the inverse Mellin transform, $\frac{\partial}{\partial T} C_H(\alpha,y,T)$ is continuous in T. □

Proposition 6.1.5

If the payoff function $H : \mathbb{R}^+ \to \mathbb{R}$ is Lipschitz and $\mathbb{E}[Y_T^2] < \infty$, then $\mathbb{E}\left[C_H(J_t, Y_t, T-t)^2\right] < \infty$, for all $t \in (0,T)$.

Proof

Using Jensen's inequality and the tower property, followed by the Markov additive property,

$$\mathbb{E}\left[(C_H(J_t,Y_t,T-t))^2\right] \leq e^{-2r(T-t)}\mathbb{E}\left[\hat{\mathbb{E}}_{(J_t,\log(Y_t))}\left[H(\hat{Y}_{T-t})^2\right]\right] = e^{-2r(T-t)}\mathbb{E}\left[H(Y_T)^2\right],$$

where $\hat{Y}$ is an independent but identically distributed copy of Y and $\hat{\mathbb{E}}$ is the corresponding expectation. However, H is Lipschitz with some constant h, so

$$\mathbb{E}\left[(C_H(J_t,Y_t,T-t))^2\right] \leq h^2 e^{-2r(T-t)}\mathbb{E}\left[Y_T^2\right] + e^{-2r(T-t)}H(0),$$

hence, $C_H(J_t, Y_t, T-t)$ has second moments if Y_T does. □

Under certain conditions, the following proposition expresses the price of a European option as the solution of a PIDE with the payoff function as a boundary condition.

Proposition 6.1.6 (PIDE for European Option Prices)

Suppose that:

1. *Y_T has finite second moments;*
2. *for each $\beta \in E$,*
   (a) *either, $\sigma_\beta^2 > 0$;*

(b) or, there exists an $\epsilon \in (0,2)$ *such that* $\liminf_{r\downarrow 0} r^{\epsilon-2} \int_{-\epsilon}^{\epsilon} |x^2| \mu_\beta(dx) > 0$*;*

3. *the payoff function* H *is Lipschitz, with Lipschitz constant* $h > 0$*;*

4. *there exists an* $s > 2$ *such that* $H(x) \le x^s$ *in some neighbourhood of* 0.

Then, $C_H(\alpha, y, t)$ *is twice continuously differentiable with respect to* y *and once continuously differentiable with respect to* t *in the domain* $E \times \mathbb{R}^+ \times \mathbb{R}^+$. *Moreover, it satisfies the PIDE*

$$\begin{aligned}
0 = &- rC_H(\alpha, y, t) - \partial_t C_H(\alpha, y, t) \\
&+ \partial_y C_H(\alpha, y, t) y \left(a_\alpha + \frac{\sigma_\alpha^2}{2} + \int_{\mathbb{R}} \left(e^u - 1 - u\mathbb{1}_{|u|\le 1}\right) \mu_\alpha(du) \right) \\
&+ \frac{1}{2} \partial_{y^2} C_H(\alpha, y, t) y^2 \sigma_\alpha^2 \\
&+ \int_{\mathbb{R}} \left(C_H(\alpha, ye^u, t) - C_H(\alpha, y, t) - y(e^u - 1)\partial_y C_H(\alpha, y, t)\right) \mu_\alpha(du) \\
&+ \sum_{\gamma \in E\setminus\{\alpha\}} \int_{\mathbb{R}} \left(C_H(\gamma, ye^u, t) - C_H(\alpha, y, t)\right) \nu_{\alpha,\gamma}(du),
\end{aligned} \tag{6.1.7}$$

on the domain $E \times \mathbb{R}^+ \times \mathbb{R}^+$*, with the boundary condition*

$$C_H(\alpha, y, 0) = H(y), \qquad \forall (\alpha, y) \in E \times \mathbb{R}^+. \tag{6.1.8}$$

Proof

First, fix the maturity of the option $T > 0$ and consider the result for the domain $E \times \mathbb{R}^+ \times (0, T)$. Then, since $T > 0$ is arbitrary, this can be extended to the full domain $E \times \mathbb{R}^+ \times \mathbb{R}^+$.

From condition (4) it follows that $H(0) = 0$, whilst by the Lipschitz property, $|H(x)| \le hx$, for all $x \ge 0$, where h is the Lipschitz constant. Hence, for any $a > 1$, $|x^{-(a+1)}H(x)| \le h|x|^{-a}$. For $a < s-1$, by (4), $|x^{-(a+1)}H(x)| \le x^{-a-1+s}$ in a neighbourhood of 0. Thus, for all $a \in (1, s-1)$, it follows that $x^{-(a+1)}H(x) \in L^1(\mathbb{R}^+)$ and so $\{\mathcal{M}H\}(-a)$ exists.

For $a \in (1,3)$, we have $\mathbb{E}[Y_T^{a-1}] < \infty$ by the assumption Y_T has second moments. Since $s > 2$, we can take $a \in (1,3) \cap (1, s-1)$, then the conditions of Corollary 6.1.1 hold and so $C_H(\alpha, y, T)$ is infinitely continuously differentiable as a function of $y \in \mathbb{R}^+$, for all $T > 0$ and $\alpha \in E$.

Moreover, the conditions on H of Proposition 6.1.4 are satisfied for $c = -a$, whilst the other conditions of Proposition 6.1.4 are given directly by the assumptions of Proposition 6.1.6. Consequently, $C_H(\alpha, y, T)$ is continuously differentiable with respect to T, for all $(\alpha, y, T) \in E \times \mathbb{R}^+ \times \mathbb{R}^+$.

For each $\alpha, \beta \in E$, denote by $\tilde{N}_\beta$ the compensated Poisson random measure associated with the Lévy process $\xi^{(\beta)}$. Also, let $\mathcal{M}_{\alpha,\beta}$ be the Poisson random measure associated with the jumps of ξ induced by a change in J from α to β, which has intensity $q_{\alpha,\beta}$. Then, let $\tilde{\mathcal{M}}_{\alpha,\beta}$ denote the compensated Poisson random measure associated with $\mathcal{M}_{\alpha,\beta}$ and let $\nu_{\alpha,\beta}(du)ds$ be the corresponding density.

Since $\{Y_t : t \geq 0\}$ is integrable, by an adaptation of the *semi-martingale decomposition* given in [18, pp 10],

$$\begin{aligned} Y_t - Y_0 = &\int_0^t \sigma_{J_s} Y_s dW(s) + \int_0^t Y_{s-}\left(a_{J_s} + \frac{\sigma_{J_s}^2}{2} + \int_{|u|\leq 1} (e^u - 1 - u)\,\mu_{J_{s-}}(du)\right) ds \\ &+ \int_0^t \int_{|u|\leq 1} Y_{s-}(e^u - 1)\tilde{N}_{J_{s-}}(ds, du) + \int_0^t \int_{|u|>1} Y_{s-}(e^u - 1) N_{J_{s-}}(ds, du) \\ &+ \sum_{\gamma \in E\setminus\{\alpha\}} \int_0^t \int_{\mathbb{R}} Y_{s-}(e^u - 1)\mathcal{M}_{J_{s-},\gamma}(ds, du), \end{aligned} \tag{6.1.9}$$

where $(W_s)_{s\geq 0}$ is a Brownian motion.

Consider the discounted price process $\hat{C}_H(J_t, Y_t, T-t) := e^{-rt} C_H(J_t, Y_t, T-t)$, which is a local martingale under the risk neutral measure. Then, since C_H is continuously differentiable with respect to T, Itô's Lemma can be applied to $\hat{C}_H$ to obtain, after some simplification,

$$\hat{C}_H(J_t, Y_t, T-t) - \hat{C}_H(J_0, Y_0, T) = \int_0^t a(s)ds + M_t,$$

where,

$$\begin{aligned} a(s) = &-\partial_t \hat{C}_H(J_{s-}, Y_{s-}, T-s) + \partial_x \hat{C}_H(J_{s-}, Y_{s-}, T-s) Y_{s-}\left(a_{J_{s-}} + \frac{\sigma_{J_{s-}}^2}{2}\right) \\ &+ \frac{1}{2}\partial_{x^2}\hat{C}_H(J_{s-}, Y_{s-}, T-s) Y_{s-}^2 \sigma_{J_{s-}}^2 \\ &+ \int_{|u|\geq 1} \left(\hat{C}_H(J_{s-}, Y_{s-}e^u, T-s) - \hat{C}_H(J_{s-}, Y_{s-}, T-s)\right) \mu_{J_{s-}}(du) \\ &+ \int_{|u|<1} \left(\hat{C}_H(J_{s-}, Y_{s-}e^u, T-s) - \hat{C}_H(J_{s-}, Y_{s-}, T-s) - uY_{s-}\partial_x \hat{C}_H(J_{s-}, Y_{s-}, T-s)\right) \mu_{J_{s-}}(du) \\ &+ \sum_{\gamma \in E\setminus\{J_{s-}\}} \int_{\mathbb{R}} \left(\hat{C}_H(\gamma, Y_{s-}e^u, T-s) - \hat{C}_H(J_{s-}, Y_{s-}, T-s)\right) \nu_{J_{s-},\gamma}(du) \end{aligned} \tag{6.1.10}$$

and

$$
\begin{aligned}
M(t) = \int_0^t & \sigma Y_{s-}\partial_x \hat{C}_H(J_{s-}, Y_{s-}, T-s)dB_s \\
& + \int_0^t \int_{\mathbb{R}} \left(\hat{C}_H(Y_{s-}e^u) - \hat{C}_H(J_{s-}, Y_{s-}, T-s)\right) \tilde{N}_{J_{s-}}(ds, du) \\
+ & \sum_{\gamma \in E\setminus\{J_{s-}\}} \int_0^t \int_{\mathbb{R}} \left(\hat{C}_H(\gamma, Y_{s-}e^u, T-s) - \hat{C}_H(J_{s-}, Y_{s-}, T-s)\right) \tilde{\mathcal{M}}_{J_{s-},\gamma}(ds, du).
\end{aligned}
$$

We now show that $M(t)$ is a *square integrable martingale*. Suppose $\alpha, \beta \in E$ and $x, y \in \mathbb{R}^+$. Then, for $s \in [0, T]$, by the Lipschitz property,

$$
\begin{aligned}
|\hat{C}_H(\alpha, x, s) - \hat{C}_H(\beta, y, s)| &\leq |\mathbb{E}_\alpha[H(xY_s)] - \mathbb{E}_\beta[H(yY_s)]| \\
&\leq |\mathbb{E}_\alpha\left[H(xY_s) - H(yY_s)\right] + \mathbb{E}_\alpha\left[H(yY_s)\right] - \mathbb{E}_\beta\left[H(yY_s)\right]| \\
&\leq h|x-y|\mathbb{E}_\alpha\left[Y_s\right] + h|y|\mathbb{E}_\alpha\left[Y_s\right] + h|y|\mathbb{E}_\beta\left[Y_s\right] \\
&\leq h\left(|x-y| + 2|y|\right)\max_{\gamma\in E}\mathbb{E}_\gamma\left[Y_s\right].
\end{aligned}
$$

Hence, for any $t \in [0, T]$ and Poisson random measure μ_t, such that

$$
\int_{\mathbb{R}} (e^u - 1)^2 \mu_t(du) < \infty, \tag{6.1.11}
$$

it follows that

$$
\begin{aligned}
&\mathbb{E}\left[\int_0^t \int_{\mathbb{R}} \left(\hat{C}_H(J_s, Y_{s-}e^u, s) - \hat{C}_H(J_{s-}, Y_{s-}, s)\right)^2 \mu_{s-}(ds, du)\right] \\
&\quad\leq \mathbb{E}\left[\int_0^t \int_{\mathbb{R}} \left(h(|e^u - 1| + 2)Y_{s-}\left(\max_{\alpha\in E}\mathbb{E}_\alpha\left[|Y_{T-s}|\right]\right)\right)^2 \mu_{s-}(ds, du)\right] \\
&\quad\leq h^2\mathbb{E}\left[\int_0^t (Y_{s-})^2 \left(\max_{\alpha\in E}\mathbb{E}_\alpha\left[|Y_{T-s}|\right]\right)^2 \int_{\mathbb{R}} \left(|e^u - 1| + 2\right)^2 \mu_{s-}(du)ds\right] \\
&\quad< \infty,
\end{aligned}
$$

provided that Y has finite first and second moments. By Theorem 3.1.1, since Y_T has finite second moments, the measures μ_α and $\nu_{\alpha,\beta}$ satisfy (6.1.11) for all $\alpha, \beta \in E$. Hence, by [3, pp 224, Chapter 4, Theorem 4.2.3] or [52, Chapter 8, Proposition 8.8], the compensated Poisson terms of M are square integrable martingales.

For the Brownian integral in M, consider $\partial_y \hat{C}_H$. For all $(\alpha, y, s) \in E \times \mathbb{R}^+ \times [0, T]$, from Lipschitz continuity, $\left|\partial_y \hat{C}_H(\alpha, y, s)\right| \leq h\mathbb{E}_\alpha\left[Y_s\right]$. Then, since Y is square integrable,

$$
\mathbb{E}\left[\int_0^t \left(\sigma_{J_{s-}} Y_{s-}\partial_x \hat{C}_H(J_{s-}, Y_{s-}, T-s)\right)^2 ds\right] \leq \mathbb{E}\left[\int_0^t \left(\sigma_{J_{s-}} Y_{s-} h\mathbb{E}_{J_{s-}}\left[Y_{T-s}\right]\right)^2 ds\right] < \infty
$$

and hence, by Itô's isometry, $\int_0^t \sigma_{J_{s-}} Y_{s-} \partial_x \hat{C}_H(J_{s-}, Y_{s-}, T-s) dB_s$ is a square integrable martingale.

Therefore, the process $M(t)$ is itself a square integrable martingale. Thus, $\int_0^t a(s)ds = \hat{C}_H(J_t, Y_t, T-t) - \hat{C}_H(J_0, Y_0, T) - M(t)$ is a local martingale. However, since this is an integral against ds, it is a continuous process with finite variation and must therefore be constant. Hence, $a(t) = 0$ for all $t \in [0, T]$.

Then, using

$$\partial_t \hat{C}_H(\alpha, y, t) = \partial_t e^{-r(T-t)} C_H(\alpha, y, t) = re^{-r(T-t)} C_H(\alpha, y, t) + e^{-r(T-t)} \partial_t C_H(\alpha, y, t),$$

making the substitution $t = T - s$ and multiplying by e^{rs} in (6.1.10) gives the required PIDE. $\square$

Remark 6.1.3

The payoff function, $H : \mathbb{R}^+ \to \mathbb{R} : x \to (x-k)^+$, of a European call option, with strike $k > 0$, satisfies the assumptions (3) and (4) of Proposition 6.1.6. Hence, if the price process corresponds to a MAP satisfying assumptions (1) and (2), then Proposition 6.1.6 can be used to price European call options.

Remark 6.1.4

It is possible to recover the Mellin transform expression of Proposition 6.1.2 from (6.1.7). By taking the Mellin transform of (6.1.7) and using the results:

$$\begin{aligned}
\frac{\partial}{\partial t}\left\{\mathcal{M}_y C_H(\alpha, y, t)\right\}(s) &= \left\{\mathcal{M}_y \frac{\partial}{\partial t} C_H(\alpha, y, t)\right\}(s), \\
\left\{\mathcal{M}_y y \frac{\partial}{\partial y} C_H(\alpha, y, t)\right\}(s) &= -s\left\{\mathcal{M}_y C_H(\alpha, y, t)\right\}(s), \\
\left\{\mathcal{M}_y y^2 C_H(\alpha, y, t)\right\}(s) &= s(s+1)\left\{\mathcal{M}_y C_H(\alpha, y, t)\right\}(s), \\
\left\{\mathcal{M}_y C_H(\alpha, ye^u, t)\right\}(s) &= e^{-us}\left\{\mathcal{M}_y C_H(\alpha, y, t)\right\}(s),
\end{aligned}$$

we obtain the equation:

$$\begin{aligned}
\partial_t \left\{\mathcal{M}_y C_H(\alpha, y, t)\right\}(s) = \left\{\mathcal{M}_y C_H(\alpha, y, t)\right\}(s) \Bigg(&-r - sa_\alpha + s\int_{\mathbb{R}} u \mathbb{1}_{\{|u|\leq 1\}} \mu_\alpha(du) \\
&+ \frac{1}{2}\sigma_\alpha^2 s^2 + \int_{\mathbb{R}} \left(e^{-us} - 1\right)\mu_\alpha(du) + \sum_{\gamma \in E\setminus\{\alpha\}} \int_{\mathbb{R}} (-1)\,\nu_{\alpha,\gamma}(du) \Bigg) \\
&+ \sum_{\gamma \in E\setminus\{\alpha\}} \left\{\mathcal{M}_y C_H(\gamma, y, t)\right\}(s) \int_{\mathbb{R}} e^{-us} \nu_{\alpha,\gamma}(du).
\end{aligned}$$

However, $\int_{\mathbb{R}} \nu_{\alpha,\gamma}(du) = q_{\alpha,\gamma}$ and $\sum_{\gamma \in E \setminus \{\alpha\}} \int_{\mathbb{R}} \nu_{\alpha,\gamma}(du) = q_\alpha$. Moreover, $\int_{\mathbb{R}} e^{-us} \nu_{\alpha,\gamma}(du) = q_{\alpha,\gamma} G_{\alpha,\gamma}(-s)$. Thus, if $m(s,t) := \left(\{\mathcal{M}_y C_H(\alpha, y, t)\}(s)\right)_{\alpha \in E}$ is a column vector of dimension $|E|$, then,

$$\partial_t m(s,t) = (-rI + F(-s))\, m(s).$$

Hence, this equation has the solution

$$m(s,t) = e^{-rt} e^{tF(-s)} m(s,0),$$

where the initial conditions give $m(s,0) = \{\mathcal{M}H(y)\}(s)(1, \cdots, 1)^T$. Thus, the Mellin transform of the option price is given by

$$\{\mathcal{M}_y C_H(\alpha, y, t)\} = e^{-rt} \{\mathcal{M}H\}(s) \sum_{\beta \in E} \left(e^{tF(-s)}\right)_{\alpha,\beta},$$

which is the relation of Proposition 6.1.2.

6.1.3 Examples: $|E| = 2$

The following examples consider the case $|E| = 2$, say $E = \{+, -\}$. In this case, the *characteristic equation* of $F(z)$ can be solved analytically to obtain the eigenvalues

$$\alpha(z) := \frac{1}{2}\left(\psi(z) + \Delta(z)\right) \qquad \text{and} \qquad \beta(z) := \frac{1}{2}\left(\psi(z) - \Delta(z)\right), \tag{6.1.12}$$

where $\psi_\pm^{(q)}(z) := \psi_\pm(z) - q_\pm$, $\psi(z) := \psi_+^{(q)}(z) + \psi_-^{(q)}(z)$ and

$$\Delta(z) := \sqrt{\left(\psi_+^{(q)}(z) - \psi_-^{(q)}(z)\right)^2 + 4q_- q_+ G_+(z) G_-(z)}.$$

Let $p(x) = (x - t\alpha)(x - t\beta)$ be the characteristic polynomial of tF. Then, by considering the remainder on division by p, there exists a convergent series, $q(x)$, and a polynomial of degree 1, $r(x) := s_0 + s_1 x$, such that $e^x = q(x)p(x) + r(x)$. Evaluating this at the eigenvalues, $t\alpha$ and $t\beta$, of tF gives

$$e^{t\alpha} = 0 + s_0 + s_1 t\alpha \qquad \text{and} \qquad e^{t\beta} = 0 + s_0 + s_1 t\beta.$$

This system of equations can be solved to obtain

$$s_0(t,z) := \frac{\alpha e^{\beta t} - \beta e^{\alpha t}}{\alpha - \beta} \qquad \text{and} \qquad s_1(t,z) := \frac{e^{\alpha t} - e^{\beta t}}{(\alpha - \beta)t}.$$

From the *Cayley-Hamilton theorem* $p(tF) = 0$, hence $e^{tF(z)} = s_0(z)I + s_1(z)tF(z)$. Then, using (6.1.12), the matrix exponential $e^{tF(z)}$ is given by

$$e^{F(z)t} = e^{\frac{t}{2}\psi(z)} \cosh\left(\frac{t}{2}\Delta(z)\right) I + \frac{e^{\frac{t}{2}\psi(z)} \sinh\left(\frac{t}{2}\Delta(z)\right)}{\Delta(z)} \begin{pmatrix} \psi_+^{(q)}(z) - \psi_-^{(q)}(z) & 2q_+ G_+(z) \\ 2q_- G_-(z) & \psi_-^{(q)}(z) - \psi_+^{(q)}(z) \end{pmatrix}, \tag{6.1.13}$$

where I is the identity matrix. This can be used to obtain explicit expressions for the Mellin transform of option prices.

Example 6.1.1

Let $\lambda_+, \lambda_- > 0$, $U_\pm \sim \text{Exp}(\lambda_\pm)$ and $\mu_\pm(dz) := q_\pm \lambda_\pm e^{-\lambda_\pm z} dz$. Also set $a_\pm = \sigma_\pm = 0$ and $q_+ = q_- =: q > 0$. Then, the MAP (J, ξ) corresponds to a *Markov modulated compound Poisson process*, of rate q, where the jumps are exponentially distributed with a rate λ_α, determined by the state of J. Hence, for each $\alpha \in \{+,-\}$ and $z \in \mathbb{C}$, with $\Re(z) < \lambda_\alpha$,

$$\begin{aligned} G_\alpha(z) &= \int_0^\infty e^{zu} \lambda_\alpha e^{-\lambda_\alpha u} du = \frac{\lambda_\alpha}{\lambda_\alpha - z}, \\ \psi_\alpha(z) &= \int_0^\infty \left(e^{uz} - 1\right) \mu_\alpha(dz) = \int_0^\infty \left(e^{uz} - 1\right) q \lambda_\alpha e^{-\lambda_\alpha u} du = q\left(G_\alpha(z) - 1\right). \end{aligned}$$

Substituting this into (6.1.13) yields

$$e^{tF(z)} = \frac{e^{-2qt}}{(G_+(z) + G_-(z))} \left[e^{qt(G_+(z)+G_-(z))} \begin{pmatrix} G_+(z) & G_+(z) \\ G_-(z) & G_-(z) \end{pmatrix} + \begin{pmatrix} G_-(z) & -G_+(z) \\ -G_-(z) & G_+(z) \end{pmatrix} \right].$$

Using Proposition 6.1.1, for each $\alpha \in E$, the price of a European call option is given by

$$C_\alpha(k) = \frac{1}{2\pi i} \int_{c+1+i\mathbb{R}} \frac{k^{-s+1}}{(s-1)s} \frac{2e^{-2qt} G_\alpha e^{qt(G_+ + G_-)}}{(G_+ + G_-)} ds + \frac{1}{2\pi i} \int_{c+1+i\mathbb{R}} \frac{k^{-s+1} e^{-2qt}}{s(s-1)} \left(\frac{G_{-\alpha} - G_\alpha}{G_+ + G_-} \right) ds,$$

for $c \in (0, \min(\lambda_+, \lambda_-))$. For $\alpha \in E$, define the function

$$R_\alpha := \delta_1(k) + \sqrt{qt\lambda_\alpha} \mathbb{1}_{\{k \ge 1\}} k^{-\lambda_\alpha} \frac{I_1\left(2\sqrt{qt\lambda_\alpha \log(k)}\right)}{\sqrt{\log(k)}},$$

where I_1 is the *modified Bessel function* of the first kind (see Appendix A.2), and

$$D_\alpha := \begin{cases} d_1^{(\alpha)} k + d_2^{(\alpha)}, & \text{if } k < 1; \\ d_3^{(\alpha)} k^{c_\alpha}, & \text{if } k \ge 1; \end{cases}$$

where,

$$d_1^{(\alpha)} := -\frac{1}{2} e^{-2qt}, \qquad d_2^{(\alpha)} := \frac{e^{-2qt} \lambda_\alpha (\lambda_{-\alpha} - 1)}{2\lambda_+ \lambda_- - (\lambda_+ + \lambda_-)}, \qquad d_3^{(\alpha)} := \frac{e^{-2qt} (\lambda_\alpha - \lambda_{-\alpha})}{2\left(2\lambda_+ \lambda_- - (\lambda_+ + \lambda_-)\right)}$$

and

$$c_\alpha = 1 - \frac{2\lambda_+ \lambda_-}{\lambda_+ + \lambda_-}.$$

Then, by applying the Mellin inversion theorem, it is shown in Appendix B.1 that

$$C_\alpha(k) = 2\{D_\alpha * R_+ * R_-\}(k) + D_{-\alpha}(k) - D_\alpha(k), \tag{6.1.14}$$

where $*$ denotes a *multiplicative convolution* (see Appendix A.3). In Appendix B.2, it is shown that C_α can then be written in series form as

$$C_\alpha(k) = \sum_{m=0}^{\infty} \frac{qt\lambda_+}{m!(m+1)!} \left(F(m, d_1^\alpha, \lambda_+, 1, k) + F(m, d_2^\alpha, \lambda_+, 0, k) + f(m, d_3^\alpha, \lambda_+, c_\alpha, k)\right)$$
$$+ \sum_{m=0}^{\infty} \frac{qt\lambda_-}{m!(m+1)!} \left(F(m, d_1^\alpha, \lambda_-, 2, k) + F(m, d_2^\alpha, \lambda_-, 1, k) + f(m, d_3^\alpha, \lambda_-, c_\alpha + 1, k)\right)$$
$$+ \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{r=0}^{\infty} 2\left(g(n, m, r, d_1^{(\alpha)}, \lambda_- + 1, 1, k) + g(n, m, r, d_2^{(\alpha)}, \lambda_-, 0, k) + g(n, m, r, d_3^{(\alpha)}, \lambda_-, \lambda_-, k)\right)$$
$$+ \mathbb{1}_{\{k<1\}}\left(d_1^{-\alpha} + d_2^{-\alpha}\right) + \mathbb{1}_{\{k\geq 1\}}\left(d_3^{-\alpha} k^{c-\alpha}\right),$$

where,

$$F(m, d, \lambda, c, k) := \frac{dk^c \Gamma(m+1, (\lambda + c + 1)\log(k \vee 1)))}{(\lambda_\alpha + c + 1)^{m+1}},$$
$$f(m, d, \lambda, c, k) := \frac{dk^c \gamma(m+1, (\lambda + c + 1)\log(k \vee 1)))}{(\lambda_\alpha + c + 1)^{m+1}},$$

$$g(n, m, r, d, l, c, k) = \frac{(qt)^{m+n+2}\sqrt{\lambda_+\lambda_-}(\lambda_- - \lambda_+)^r (r+m)!}{m!(m+1)!(n+1)!r!(r+m+n+1)!} \frac{dk^c}{l^{r+m+n+2}} \Gamma(r+m+n+2, l\log(k)),$$

and $\Gamma(\cdot,\cdot)$ and $\gamma(\cdot,\cdot)$ denote the upper and lower incomplete Gamma functions, respectively, (see Appendix A.1).

Now consider evaluating this at $k = 1$, the so called "at the money" option. In this case, the triple convolution becomes

$$\left(D_\alpha * J_+ * J_-\right)(1) = qt\sqrt{\lambda_+\lambda_-} \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{r=0}^{\infty} \frac{qt^{m+n+1}(\lambda_- - \lambda_+)^r (r+m)!}{m!(m+1)!(n+1)!r!(r+m+n+1)!}$$
$$\times \left(\frac{d_1^{(\alpha)}}{(\lambda_- + 1)^{r+m+n+2}} \Gamma(r+m+n+2) + \frac{d_2^{(\alpha)}}{(\lambda_-)^{r+m+n+2}} \Gamma(r+m+n+2)\right),$$

where the upper incomplete Gamma functions have become complete Gamma functions and the lower incomplete Gamma functions have evaluated to 0. Then, first considering the sum over n,

$$\left(D_\alpha * J_+ * J_-\right)(1) = qt\sqrt{\lambda_+\lambda_-} \sum_{m=0}^{\infty}\sum_{r=0}^{\infty} \frac{(qt)^m (r+m)!(\lambda_- - \lambda_+)^r}{m!(m+1)!r!}$$
$$\times \left(\frac{d_1^{(\alpha)}}{(\lambda_- + 1)^{r+m+1}} \left(\exp\left(\frac{qt}{\lambda_- + 1}\right) - 1\right) + \frac{d_2^{(\alpha)}}{\lambda_-^{r+m+1}} \left(\exp\left(\frac{qt}{\lambda_-}\right) - 1\right)\right).$$

Now, considering the sum over m and identifying the *hypergeometric function* ${}_1F_1$ (see

Appendix A.2), gives

$$
\begin{aligned}
\left(D_\alpha * J_+ * J_-\right)(1) = {} & qt\sqrt{\lambda_+\lambda_-}d_1^{(\alpha)}\left(\exp\left(\frac{qt}{\lambda_-+1}\right)-1\right)\sum_{r=0}^{\infty}\frac{1}{\left(\lambda_-+1\right)^{r+1}}\,{}_1F_1\left(r+1,2;\frac{qt}{\lambda_-+1}\right)\\
& + qt\sqrt{\lambda_+\lambda_-}d_2^{(\alpha)}\left(\exp\left(\frac{qt}{\lambda_-}\right)-1\right)\sum_{r=0}^{\infty}\frac{1}{\lambda_-^{r+1}}\,{}_1F_1\left(r+1,2;\frac{qt}{\lambda_-}\right).
\end{aligned}
$$

Moreover, for each $\alpha \in E$ and $c_\alpha \in \mathbb{R}$

$$
\begin{aligned}
\sum_{m=0}^{\infty}\frac{qt\lambda_\beta}{m!(m+1)!}F(m,d^{(\alpha)},\lambda_\beta+c_\alpha+1,c_\alpha,1) &= \sum_{m=0}^{\infty}\frac{qt\lambda_\beta}{m!(m+1)!}\frac{d^{(\alpha)}k^{c_\alpha}m!}{(\lambda_\beta+c_\alpha+1)^{m+1}}\\
&= qtd^{(\alpha)}\lambda_\beta\left(\exp\left(\frac{1}{\lambda_\beta+c_\alpha+1}\right)-1\right),
\end{aligned}
$$

hence,

$$
\left\{\left\{\mathbb{1}_{\{x\geq 1\}}\frac{1}{x}J_\beta(x)\right\} * D_\alpha(x)\right\}(1) = qtd^{(\alpha)}\lambda_\beta\left(\exp\left(\frac{1}{\lambda_\beta+2}+\frac{1}{\lambda_\beta+1}+\frac{1}{\lambda_\beta+c_\alpha+1}\right)-3\right).
$$

Combining these results,

$$
\begin{aligned}
C_\alpha(1) = {} & 2qt\sqrt{\lambda_+\lambda_-}d_1^{(\alpha)}\left(\exp\left(\frac{qt}{\lambda_-+1}\right)-1\right)\sum_{r=0}^{\infty}\frac{1}{\left(\lambda_-+1\right)^{r+1}}\,{}_1F_1\left(r+1,2;\frac{qt}{\lambda_-+1}\right)\\
& + qt\sqrt{\lambda_+\lambda_-}d_2^{(\alpha)}\left(\exp\left(\frac{qt}{\lambda_-}\right)-1\right)\sum_{r=0}^{\infty}\frac{1}{\lambda_-^{r+1}}\,{}_1F_1\left(r+1,2;\frac{qt}{\lambda_-}\right)\\
& + qtd^{(\alpha)}\lambda_+\left(\exp\left(\frac{1}{\lambda_++2}+\frac{1}{\lambda_++1}+\frac{1}{\lambda_++c_\alpha+1}\right)-3\right)\\
& + qtd^{(\alpha)}\lambda_-\left(\exp\left(\frac{1}{\lambda_-+3}+\frac{1}{\lambda_-+2}+\frac{1}{\lambda_-+c_\alpha+2}\right)-3\right)\\
& + k^{c_{-\alpha}}d_3^{(-\alpha)}.
\end{aligned}
$$

Example 6.1.2

Simpler results can be found by continuing Example 4.3.1, where $(1,1)^T$ is a right eigenvector of $F(x)$, for all $x > 0$. Then, if (4.3.1) is satisfied, from Proposition 6.1.1,

$$
\{\mathcal{M}_k C_\alpha(k)\}(u) = e^{-rT}\frac{\exp\left(T\left(\psi_+(u+1)+q_+\left(G_+(u+1)-1\right)\right)\right)}{u(u+1)}.
$$

In the particular case of Example 4.3.1, this gives

$$
\{\mathcal{M}_k C_\alpha(k)\}(u) = e^{-rT}\exp\left(-\frac{2qT(u+1)}{u+2}\right)\frac{1}{u(u+1)} = e^{-(r+2q)T}\exp\left(\frac{2qT}{u+2}\right)\frac{1}{u(u+1)}.
$$

Define the function $R : \mathbb{R}^+ \to \mathbb{R}$ by

$$
R(k) := \delta_1(k) + \sqrt{2qT}\mathbb{1}_{\{k\leq 1\}}k^2\frac{I_1\left(2\sqrt{-2qT\log(k)}\right)}{\sqrt{-\log(k)}}, \qquad k \geq 0.
$$

Then, it is known that

$$\{\mathcal{M}R\}(u) = \exp\left(\frac{2qT}{2+u}\right) \qquad \text{and} \qquad \mathcal{M}\left\{(1-k)^+\right\}(s) = \frac{1}{u(u+1)}.$$

Hence, by the Mellin inversion theorem,

$$C_\alpha(k) = e^{-(r+2q)T}\left\{R(x) * (1-x)^+\right\}(k),$$

where $*$ denotes the Mellin type convolution. Expanding this convolution gives

$$\begin{aligned}
&\left\{R(x^{-1}) * (1-x)^+\right\}(k)\\
&= \int_0^\infty \delta_1(x)\left(1-\frac{k}{x}\right)^+\frac{1}{x} + \sqrt{2qT}\mathbb{1}_{\{x\le 1\}}x^2\frac{I_1\left(2\sqrt{-2qT\log(x)}\right)}{\sqrt{-\log(x)}}\left(1-\frac{k}{x}\right)^+\frac{1}{x}dx\\
&= (1-k)^+ + \sqrt{2qT}\int_k^1 (x-k)\frac{I_1(2\sqrt{-2qT\log(x)})}{\sqrt{-\log(x)}}dx,
\end{aligned}$$

whenever $k \le 1$, whilst $\left\{R(x^{-1}) * (1-x)^+\right\}(k) = 0$ for $k > 1$. However, using the series expansion of I_1 and a change of variables,

$$\int_k^1 \frac{I_1(2\sqrt{-2qT\log(x)})}{\sqrt{-\log(x)}}dx = \sum_{m=0}^\infty \frac{(2qT)^{m+1/2}}{m!(m+1)!}\gamma(m+1,\log(1/k))$$

and

$$\int_k^1 \frac{xI_1(2\sqrt{-2qT\log(x)})}{\sqrt{-\log(x)}}dx = \sum_{m=0}^\infty \frac{2^{-1/2}(qT)^{m+1/2}}{m!(m+1)!}\gamma(m+1,2\log(1/k)).$$

Hence, for all $k \le 1$,

$$\begin{aligned}
C_\alpha(k) = e^{-(r+2q)T}\Bigg(1-k+\sum_{m=0}^\infty &\frac{(qT)^{m+1/2}2^{-1/2}\gamma(m+1,2\log(1/k))}{m!(m+1)!}\\
&-\frac{k(2qT)^{m+1/2}\gamma(m+1,\log(1/k))}{m!(m+1)!}\Bigg)
\end{aligned} \tag{6.1.15}$$

and $C_\alpha(k) = 0$ for all $k > 1$. Notice that $C_\alpha(1) = 0$, hence the option price is continuous at this transition point. Moreover, it is not surprising that $C_\alpha(k) = 0$ for $k > 1$, since the ξ_t is a (weakly) decreasing processes, thus once $\xi < k$ the option can never regain its value. The maximal value of the call option is achieved when $k = 0$. In this case,

$$C_\alpha(0) = \left(1-\frac{1}{\sqrt{2qT}}\right)e^{-(r+2q)T} + \frac{1}{\sqrt{2qT}}e^{-(r+q)T}.$$

Whilst expressions for C_α were obtained in some of these examples, albeit with high levels of complexity, the main benefit of the Mellin transform approach is that it allows numerical computation of option prices via the *Fast Fourier Transform*. We can also use the Mellin transform expression to conduct sensitivity analysis of option prices.

6.2 Asian Option Pricing

An *Asian option*, with *payoff function* $H : \mathbb{R}^+ \to \mathbb{R}$ and *maturity* $T \geq 0$, on an asset with price process $\{Y_t : t \geq 0\}$, is a contract which pays its owner $\int_{T_0}^{T} Y_s ds$ at time T, for some $T_0 \in (0, T)$. Similarly to the European call or put, an Asian option with payoff function $H(x) := (x-k)^+$, for some $k > 0$, is called an *Asian call option*, whilst if the payoff function is $H(x) := (k-x)^+$, then it is called an *Asian put option*. In both cases, k is referred to as the *strike price*.

Under an equivalent martingale measure, $\mathbb{P}$, the price of an Asian option at time $t < T$ is given by

$$e^{-r(T-t)}\mathbb{E}\left[H\left(\int_{T_0}^{T} Y_s \, ds \right) \middle| \mathcal{F}_t \right].$$

As in Section 6.1, assume that the price process of the underlying asset is given by an exponential MAP model. That is, $Y_t := \exp(\xi_t)$, for all $t \geq 0$, where (J, ξ) is a MAP. Then, following the simplifying steps of [26], for $t \in (T_0, T)$,

$$\mathbb{E}\left[H\left(\int_{T_0}^{T} Y_s \, ds \right) \middle| \mathcal{F}_t \right] = \mathbb{E}\left[H\left(\int_{T_0}^{t} e^{\xi_s} \, ds + e^{\xi_t} \int_{0}^{T-t} e^{\xi_{t+s}-\xi_t} \, ds \right) \middle| \mathcal{F}_t \right].$$

Since $\gamma_t := \int_{T_0}^{t} e^{\xi_s} ds$ and e^{ξ_t} are both $\mathcal{F}_t$ measurable, there is a function $H_t(x) := H(\gamma_t + e^{\xi_t} x)$, such that

$$\mathbb{E}\left[H\left(\int_{T_0}^{T} Y_s \, ds \right) \middle| \mathcal{F}_t \right] = \mathbb{E}\left[H_t\left(\int_{0}^{T-t} e^{\xi_{t+s}-\xi_t} \, ds \right) \middle| \mathcal{F}_t \right] = \mathbb{E}_{J_t}\left[H_t\left(\int_{0}^{T-t} e^{\hat{\xi}_s} \, ds \right) \right],$$

where $\hat{\xi}$ is an independent copy of ξ and the second equality follows from the Markov additive property. Hence, an understanding of the price of an Asian option can be obtained by studying the simpler object

$$C_H^A(\alpha, y, T) := e^{-rT}\mathbb{E}_{\alpha, \log(y)}\left[H\left(\int_{0}^{T} e^{\xi_s} \, ds \right) \right].$$

From Chapter 4, the Mellin transform of the density of A_∞ is known for certain classes of MAP. Therefore, the Mellin transform method discussed in Section 6.1.1 can be adapted to Asian options. In the case of a meromorphic Lévy process a similar Mellin transform approach to the pricing of Asian options was considered in [28].

Proposition 6.2.1 (Asian Option Valuation)
Suppose $H : \mathbb{R}^+ \to \mathbb{R}$ and there exists $s \in \mathbb{R}$ such that $\mathbb{E}\left[A_T^{-s}\right] < \infty$ and $\{\mathcal{M}H\}(s)$ exists.

Furthermore, let $(J_0, Y_0) = (\alpha, y) \in E \times \mathbb{R}^+$. Then, for all $T > 0$,

$$C_H^A(\alpha, y, T) = \mathcal{L}_q^{-1}\left\{\frac{1}{q+r}\mathcal{M}_z^{-1}\left\{\{\mathcal{M}H\}(z)\mathbb{E}_\alpha\left[A_{\tau_{q+r}}^{-z}\right]\right\}(y)\right\}(T), \tag{6.2.1}$$

where τ_q is an exponentially distributed random variable of rate $q > 0$, independent of (J, Y) and the Mellin inversion is taken along the line $s + i\mathbb{R}$.

Remark 6.2.1

If the MAP satisfies one of the sets of assumptions from Section 4.2, then $\mathbb{E}[A_{\tau_q}^{-z}]$ is known to be given by an infinite product. Moreover, both the Laplace transform and the Mellin transform can be obtained from the Fourier transform, which enables efficient numerical computation using the *Fast Fourier Transform*. Hence, the expression for $C_H^A(\alpha, y, T)$ can be numerically approximated.

Proof

For a fixed maturity $T > 0$, first consider how the random variable A_T is related to A_∞. By considering a MAP which is killed after an exponentially distributed amount of time, this can be done using a Laplace transform with respect to time. This is the same technique as was used in [26] when considering geometric Brownian motion. For this purpose, introduce a random variable τ_q, which is exponentially distributed with rate $q > 0$. Then, from the density of an exponential random variable, it follows that

$$\begin{aligned}(q+r)\mathcal{L}_T\left\{C_H^A(\alpha, y, T)\right\}(q) &= \int_0^\infty (q+r)e^{-qT}C_H^A(\alpha, y, T)dT \\ &= \mathbb{E}\left[\int_0^\infty (q+r)e^{-(q+r)T}H(A_T)dT\right] = \mathbb{E}\left[H\left(A_{\tau_{q+r}}\right)\right],\end{aligned}$$

and thus,

$$C_H^A(\alpha, y, T) = \mathcal{L}_q^{-1}\left\{(q+r)^{-1}\mathbb{E}\left[H\left(A_{\tau_{q+r}}\right)\right]\right\}(T).$$

Following the same calculation as in the European case in Proposition 6.1.2,

$$\mathcal{M}_y\left\{\mathbb{E}_{\alpha,\log(y)}\left[H\left(A_{\tau_{q+r}}\right)\right]\right\}(z) = \{\mathcal{M}H\}(z)\mathbb{E}_\alpha\left[A_{\tau_{q+r}}^{-z}\right],$$

for all $z \in \mathbb{C}$ such that $\Re(z) = s$. Thus,

$$C_H^A(\alpha, y, T) = \mathcal{L}_q^{-1}\left\{(q+r)^{-1}\mathcal{M}_z^{-1}\left\{\{\mathcal{M}H\}(z)\mathbb{E}_\alpha\left[A_{\tau_{q+r}}^{-z}\right]\right\}(y)\right\}(T).$$

$\square$

Corollary 6.2.1 (Asian Option Sensitivities)

Suppose $(\alpha, y, T) \in E \times \mathbb{R}^+ \times \mathbb{R}^+$ and the conditions of Proposition 6.2.1 hold. Then:

1. *The sensitivity of C_H^A to the maturity, T, is given by*

$$\frac{\partial}{\partial T} C_H^A(\alpha, y, T) = \left(\frac{q}{q+r}\right) \mathcal{M}_z^{-1} \left\{ \{\mathcal{M}H\}(z) \mathbb{E}_\alpha \left[A_{\tau_{q+r}}^{-z} \right] \right\} (y) - H(y),$$

2. *The n^{th}-order sensitivity to the spot price, y, is given by*

$$\frac{\partial^n}{\partial y^n} C_H^A(\alpha, y, T)$$
$$= \mathcal{L}_q^{-1} \left\{ \frac{1}{q+r} \mathcal{M}_z^{-1} \left\{ \{MH\}(z-n) \left(\prod_{k=1}^n \left(\frac{-F(n-k-z)}{(n-k-z)^2} \right)^{-1} \mathbb{E}\left[A_{\tau_q}^{-z} \right] \right)_\alpha \right\} \right\}. \tag{6.2.2}$$

Proof

(1) Time sensitivity:

The Laplace transform satisfies the differential property: $\mathcal{L}\{f'\}(s) = s\mathcal{L}F(s) - f(0^+)$. Thus, from (6.2.1),

$$\mathcal{L}_T \left\{ \frac{\partial}{\partial T} C_H^A(\alpha, y, T) \right\} (q) = q\mathcal{L}_T \left\{ C_H^A(\alpha, y, T) \right\} (q) - C_H^A(\alpha, y, 0)$$
$$= \left(\frac{q}{q+r}\right) \mathcal{M}_z^{-1} \left\{ \{\mathcal{M}H\}(z) \mathbb{E}_\alpha \left[A_{\tau_{q+r}}^{-z} \right] \right\} (y) - H(y),$$

where taking the inverse Laplace transform gives the desired result.

(2) Spot sensitivity:

Differentiating (6.2.1) with respect to y gives, for each $n \in \mathbb{N}$,

$$\frac{\partial^n}{\partial y^n} C_H^A(\alpha, y, T) = \mathcal{L}_q^{-1} \left\{ \frac{1}{(q+r)} \frac{\partial^n}{\partial y^n} \mathcal{M}_z^{-1} \left\{ \{\mathcal{M}H\}(z) \mathbb{E}_\alpha \left[A_{\tau_{q+r}}^{-z} \right] (y) \right\} \right\} (T).$$

However, using standard results for taking the Mellin transform of a derivative and the recurrence relation for $\mathbb{E}[A_{\tau_{q+r}}^{n-z}]$, gives

$$\frac{\partial^n}{\partial y^n} \mathcal{M}_z^{-1} \left\{ \{\mathcal{M}H\}(z) \mathbb{E}_\alpha \left[A_{\tau_{q+r}}^{-z} \right] \right\} (y)$$
$$= \mathcal{M}_z^{-1} \left\{ (-1)^n \left(\prod_{k=1}^n (k-z) \right) \{\mathcal{M}H\}(z-n) \mathbb{E}_\alpha \left[A_{\tau_{q+r}}^{n-z} \right] \right\} (y)$$
$$= \mathcal{M}_z^{-1} \left\{ \{\mathcal{M}H\}(z-n) \left(\prod_{k=1}^n \left(\frac{-F(n-k-z)}{(n-k-z)^2} \right)^{-1} \mathbb{E}[A_{\tau_{q+r}}^{-z}] \right)_\alpha \right\} (y),$$

which substituted into the previous result gives the equation of the corollary. □

6.3 Comparison of European and Asian Call Options

We wish to make a comparison of the prices of European and Asian call options under an exponential MAP model. To do this we will need the following Martingale properties of exponential MAPs, which we derive from *Dynkin's formula*.

Let A denote the *(extended) generator* of the Markov process (J, Y), where $Y_t := \exp(\xi_t)$ for all $t \geq 0$ and (J, ξ) is a MAP. Denote the *domain* of the generator by $D(A)$ and it's *extended domain* by $\mathbb{D}(A)$. From [11], it is known that, for a bounded continuous function $f \in D(A)$,

$$(Af)(\alpha, x) = \left(\mathcal{L}^{(\alpha)} \exp \circ f(\alpha, \cdot)\right)(\log|x|) + \sum_{\beta \in E \setminus \{\alpha\}} q_{\alpha,\beta} \left(\mathbb{E}\left[f(\beta, x \exp(U_{\alpha\beta}))\right] - f(\alpha, x)\right), \tag{6.3.1}$$

for all $(\alpha, x) \in E \times \mathbb{R}^+$, where $\mathcal{L}^{(\alpha)}$ is the generator of $\xi^{(\alpha)}$. A martingale condition for Y can now be stated.

Theorem 6.3.1 (Martingale Condition for Exponential MAPs)
Let $f : E \times \mathbb{R}^+ \to \mathbb{R} : (\alpha, x) \to x$ and $\{\mathcal{F}_t\}_{t \geq 0}$ be the natural filtration of (J, ξ). Then, Y is a martingale with respect of $\mathcal{F}$, if and only if, $f \in \mathbb{D}(A)$ and $(Af)(\alpha, 1) = 0$, for all $\alpha \in E$.

Proof

Sufficiency:

First suppose Y is a martingale and thus is integrable. Then, Theorem 3.1.1(4) holds and under these conditions a semi-martingale decomposition of Y is given by (6.1.9). This can then be rearranged to give, for all $t \geq 0$,

$$\begin{aligned} Y_t = M_t + \int_0^t Y_{s-} \Bigg(& a_{J_s} + \frac{\sigma_{J_s}^2}{2} \\ & + \sum_{\gamma \in E \setminus \{J_{s-}\}} \int_{\mathbb{R}} (e^u - 1) \nu_{J_{s-},\gamma}(du) + \int_{\mathbb{R}} \left(e^u - 1 - u \mathbb{1}_{\{|u| \leq 1\}}\right) \mu_{J_{s-}}(du) \Bigg) ds, \end{aligned}$$

where $\{M_t : t \geq 0\}$ is a martingale. However, by applying (6.3.1) to $f : E \times \mathbb{R} : (\alpha, x) \to x$, the integrand with respect to ds can be identified as $(Af)(J_{s-}, Y_{s-})$. Hence

$$Y_t = M_t + \int_0^t (Af)(J_{s-}, Y_{s-}) ds,$$

and therefore, $f \in \mathbb{D}(A)$. Then, since Y is also a martingale,

$$\int_0^t (Af)(J_s, Y_s) ds = Y_s - M_s,$$

is a martingale with respect to $\{\mathcal{F}_t\}_{t \geq 0}$. Thus, for all $t > u > 0$, first using the martingale property, then the Markov property and finally Fubini's theorem, gives that

$$0 = \mathbb{E}\left[\int_u^t (Af)(J_s, Y_s) ds \,\middle|\, \mathcal{F}_u\right] = \mathbb{E}_{J_u, \xi_u}\left[\int_0^{t-u} (Af)(\hat{J}_s, \hat{Y}_s) ds\right] = \int_0^{t-u} \mathbb{E}_{J_u, \xi_u}\left[(Af)(\hat{J}_s, \hat{Y}_s)\right] ds,$$

where $(\hat{J}, \hat{Y})$ is an independent but identically distributed copy of (J, Y). However, from the definition of f and since $(J, \log(Y))$ is a MAP, for all $(\sigma, a) \in \mathbb{R}^+ \times E$,

$$(Af)(\sigma, a) = \lim_{h \searrow 0} \frac{\mathbb{E}_{\sigma,a}[f(J(h), Y(h))] - f(\sigma, a)}{h} = \lim_{h \searrow 0} \frac{a\mathbb{E}_{\sigma,1}[Y(h)] - a}{h} = a(Af)(\sigma, 1).$$

Then, substituting this into the previous integral gives

$$0 = \int_0^{t-u} \mathbb{E}_{J_u,\xi_u} \left[\hat{Y}_s(Af)(\hat{J}_s, 1)\right] ds,$$

for all $t > u > 0$. Differentiating this with respect to t and setting $t = u + s$, gives, for $s, u > 0$,

$$0 = \mathbb{E}_{J_u,\xi_u} \left[\hat{Y}_s(Af)(\hat{J}_s, 1)\right] \qquad \text{a.e..}$$

The limit as $s \downarrow 0$ can be found by adapting part of the proof of [11, Theorem 6(i)]. Splitting around the event $\{\hat{T}_1 > s\}$ gives

$$\begin{aligned}\mathbb{E}_{J_u,\xi_u}[\hat{Y}_s(Af)(\hat{J}_s, 1)] = \mathbb{E}_{J_u,\xi_u}&[\hat{Y}_s(Af)(\hat{J}_s, 1) \mid \hat{T}_1 > s]\mathbb{P}_{J_u}(\hat{T}_1 > s) \\ &+ \mathbb{E}_{J_u,\xi_u}[\hat{Y}_s(Af)(\hat{J}_s, 1) \mid \hat{T}_1 \leq s]\mathbb{P}_{J_u}(\hat{T}_1 \leq s).\end{aligned}$$

Then, considering the first term,

$$\begin{aligned}\mathbb{E}_{J_u,\xi_u}[\hat{Y}_s(Af)(\hat{J}_s, 1) \mid T_1 > s]\mathbb{P}_{J_u}(T_1 > s) &= \mathbb{E}_{J_u,\xi_u}[\hat{Y}_0 \exp(\hat{\xi}_s^{(1)})(Af)(J_u, 1)]e^{-sq_1} \\ &= Y_u(Af)(J_u, 1)\mathbb{E}[\exp(\hat{\xi}_1^{(1)})]^s e^{-sq_1},\end{aligned}$$

so, letting $s \downarrow 0$, we obtain

$$\lim_{s \downarrow 0} \mathbb{E}_{J_u,\xi_u}[\hat{Y}_s(Af)(\hat{J}_s, 1) \mid \hat{T}_1 > s]\mathbb{P}_{J_u}(\hat{T}_1 > s) = Y_u(Af)(J_u, 1).$$

To see that the second term tends to 0 as $s \downarrow 0$, recall that Y is integrable and that $\mathbb{P}_{J_0}(T_1 \leq s) = 1 - e^{-q_1 s} \to 0$ as $s \downarrow 0$. Since the inequality

$$|Y_s(Af)(J_s, 1)\mathbb{1}_{\{T_1 \leq s\}}| \leq |Y_s| \max_{\alpha \in E} |(Af)(\alpha, 1)|,$$

holds with an integrable right hand side and $Y_s(Af)(J_s, 1)\mathbb{1}_{\{T_1 \leq s\}} \to 0$ almost surely as $s \downarrow 0$, the dominated convergence theorem yields

$$\lim_{s \downarrow 0} \mathbb{E}_{J_u,\xi_u}[\hat{Y}_s(Af)(J_s, 1); T_1 \leq s] = 0 \qquad \text{a.s..}$$

Combining these results gives

$$0 = \lim_{s \downarrow 0} \mathbb{E}_{J_u,\xi_u}[\hat{Y}_s(Af)(\hat{J}_s, 1)] = Y_u(Af)(J_u, 1) \qquad \text{a.s.}$$

and dividing by Y_u, which is non-zero, then gives $0 = (Af)(J_u, 1)$ a.e.. Since $J_u = \alpha$ with non-zero probability for any $u > 0$, $\alpha \in E$, it follows that $(Af)(\alpha, 1) = 0$, for all $\alpha \in E$.

Necessity:

Suppose that $f \in \mathbb{D}(A)$ and $(Af)(\alpha, 1) = 0$ for all $\alpha \in E$. Then, from equation (6.3.1), for all $\alpha, \beta \in E$,

$$\infty > |(\mathcal{L}^{(\alpha)} \exp)(\log(1))| = \left| a_\alpha + \frac{1}{2}\sigma_\alpha^2 + \int_{\mathbb{R}} \left(e^u - 1 - u\mathbb{1}_{|u|<1}\right) \mu_\alpha(du) \right|$$

and

$$\infty > \left| q_{\alpha,\beta}\mathbb{E}\left[1 - \exp(U_{\alpha,\beta})\right]\right|.$$

This implies that both $\xi^{(\alpha)}$ and $U_{\alpha,\beta}$ have exponential moments for all $\alpha, \beta \in E$, thus by Theorem 3.1.1, $\mathbb{E}[Y_t] < \infty$ for all $t \geq 0$.

The assumption $(Af)(\alpha, 1) = 0$, combined with the multiplicative invariance property gives, for all $t > u > 0$,

$$\int_u^t (Af)(J_s, Y_s)ds = \int_u^t Y_s(Af)(J_s, 1)ds = 0.$$

Then, by the definition of the extended generator, it follows that

$$M(t) := Y_t - Y_0 - \int_0^t (Af)(J_s, 1)ds = Y_t - Y_0,$$

is a martingale with respect to $\{\mathcal{F}_t\}_{t\geq 0}$. Thus, Y must also be a martingale. □

Using equation (6.3.1) and the Lévy-Khintchine formula to expand the generator of a MAP, the following condition for such a process to be a martingale can be found.

Corollary 6.3.1

The process Y is a martingale if and only if, for all $\alpha \in E$,

$$\sum_{\beta\in E} q_{\alpha,\beta}\mathbb{E}\left[1 - \exp(U_{\alpha,\beta})\right] = a_\alpha + \frac{1}{2}\sigma_\alpha^2 + \int_{\mathbb{R}} \left(\exp(y) - 1 - \frac{y}{1+|y|}\mu_\alpha(dy) \right) < \infty, \quad (6.3.2)$$

or equivalently, for all $\alpha \in E$,

$$\sum_{\beta\in E} q_{\alpha,\beta}\left(1 - G_{\alpha,\beta}(1)\right) = \psi_\alpha(1) < \infty. \quad (6.3.3)$$

Proof

By Theorem 6.3.1, the process Y is a martingale if and only if $f \in \mathbb{D}(A)$ and $(Af)(\alpha, 1) = 0$, for all $\alpha \in E$ when $f : \mathbb{R}^+ \times E \to \mathbb{R} : (\sigma, x) \to x$ and A is the (extended) generator of the pair (J, Y).

Suppose equation (6.3.3) holds, then for each $\alpha, \beta \in E$, both $\mathbb{E}\left[\exp(\xi_1^{(\alpha)})\right] < \infty$ and $\mathbb{E}\left[\exp\left(U_{\alpha,\beta}\right)\right] < \infty$, hence by Theorem 3.1.1, Y is integrable. Moreover, if Y is integrable then Theorem 3.1.1 gives the finiteness requirement of equation (6.3.3).

Now consider the expectation requirement for a martingale. From equation (6.3.1), the (extended) generator of (J, Y) applied to $f : \mathbb{R}^+ \times E \to \mathbb{R} : (\alpha, x) \to x$ gives

$$(Af)(\alpha, x) = \mathcal{L}^{(\alpha)}(\exp)(\log x) + \sum_{\beta \in E} x q_{\alpha,\beta} \mathbb{E}\left[\exp(U_{\alpha,\beta}) - 1\right],$$

where $\mathcal{L}^{(\alpha)}$ is the extended generator of the Lévy process $\xi^{(\alpha)}$.

Thus, the condition $(Af)(\alpha, 1) = 0$ is equivalent to

$$\sum_{\beta \in E} q_{\alpha,\beta} \left(1 - \mathbb{E}[\exp(U_{\alpha,\beta})]\right) = a_\alpha + \frac{1}{2}\sigma_\alpha^2 + \int_{\mathbb{R}} \left(\exp(y) - 1 - y\mathbb{1}_{\{|y| \le 1\}}\right) \mu_\alpha(dy) = \psi_\alpha(1).$$

$\square$

Example 6.3.1 (Spectrally Negative Lévy Process)
As an example, suppose that X is a *spectrally negative Lévy process*, so that the (positive) Laplace exponent, $\psi(z)$, is defined for all $z \in \mathbb{R}^+$. Let the characteristic triplet be (a_X, σ_X, μ_X). Then, by [36, pp 82] the process $\{\xi_t := X_t - \psi(1)t : t \ge 0\}$ is also a Lévy process and has characteristic triplet $(a_\xi, \sigma_\xi, \mu_\xi)$, given by

$$a_\xi := a_X - \psi(1)t, \qquad \sigma_\xi := \sigma_X \qquad \text{and} \qquad \mu_\xi := \mu_X.$$

Moreover, from [36, pp 82] we know that the process $\{Y_t := \exp(\xi_t) : t \ge 0\}$ is a martingale and (J, Y) is a MAP for any constant Markov chain J. Hence, we can check the conditions of Corollary 6.3.1 are satisfied. In particular, because the Markov chain J is constant, the left-hand side of Corollary 6.3.1 is 0. The right-hand side is given by

$$\begin{aligned} a_\xi + \frac{\sigma_\xi^2}{2} + \int_{-\infty}^0 \left(\exp(y) - 1 - \frac{y}{1+|y|}\right) \mu_\xi(dy) \\ = a_X - \left(a_X + \frac{\sigma_X^2}{2} + \int_{-\infty}^0 \left(\exp(y) - 1 - \frac{y}{1+|y|}\right) \mu_X(dy)\right) \\ + \frac{\sigma_X^2}{2} + \int_{-\infty}^0 \left(\exp(y) - 1 - \frac{y}{1+|y|}\right) \mu_X(dy) = 0 \end{aligned}$$

and therefore Corollary 6.3.1 is satisfied.

The martingale result can be extended to obtain *sub/super-martingale* conditions for Y. For convenience, if A is the (extended) generator of a process and $f : E \times \mathbb{R} \to \mathbb{R} : (\sigma, x) \to x$, introduce the notation $A^{(\alpha)} := (Af)(\alpha, 1)$, for all $\alpha \in E$.

The following theorem considers call options which have a payoff function $H : \mathbb{R} \to \mathbb{R} : x \to (x - k)^+$ for some strike price $k \in \mathbb{R}^+$.

Proposition 6.3.1 (Sub/Super Martingale Conditions for Lamperti-Kiu Processes)
An exponential MAP, with (extended) generator A, is a sub-martingale if and only if $0 \leq A^{(\alpha)} < \infty$, for all $\alpha \in E$, and is a super-martingale if and only if $-\infty < A^{(\alpha)} \leq 0$, for all $\alpha \in E$.

Proof
Similarly to the martingale property in Theorem 6.3.1, integrability in both cases is equivalent to $|A^{(\alpha)}| < \infty$ for all $\alpha \in E$. This is obtained by considering the decomposition of the (extended) generator given in equation (6.3.1) and comparing it to Theorem 3.1.1.

Now consider the expectation properties. First, suppose that $A^{(\alpha)} \geq 0$ for each $\alpha \in E$. Then, by the semi-martingale decomposition from [18, Section 2.1, pp 10], used in the proof of Theorem 6.3.1,

$$M_t := Y_t - Y_0 - \int_0^t (Af)(J_s, Y_s) ds,$$

is a martingale with respect to $\{\mathcal{F}_t\}_{t \geq 0}$. For $0 < s < t$, by letting $(\hat{J}, \hat{Y})$ denote an independent but identically distributed copy of (J, Y),

$$\mathbb{E}[Y_t \mid \mathcal{F}_s] - Y_s = \mathbb{E}\left[\int_s^t (Af)(J_u, Y_u) du \,\middle|\, \mathcal{F}_s \right] = \int_0^{t-s} \mathbb{E}_{J_s, Y_s}[(Af)(\hat{J}_u, \hat{Y}_u)] du.$$

Applying multiplicative invariance to the generator of (J, Y) gives

$$\int_0^{t-s} \mathbb{E}_{J_s, Y_s} \left[(Af)(\hat{J}_u, \hat{Y}_u) \right] du = \int_0^{t-s} \mathbb{E}_{J_s, Y_s} \left[\hat{Y}_u (Af)(\hat{J}_u, 1) \right] du \geq 0,$$

where the last inequality is due to the assumption that $A^{(\alpha)} \geq 0$ for all $\alpha \in E$. Hence $\mathbb{E}[Y_t \mid \mathcal{F}_s] \geq Y_s$ and so Y is a sub-martingale.

Conversely, suppose that Y is a sub-martingale. Then, by integrability of sub-martingales, it follows that $|A^{(\alpha)}| < \infty$, for all $\alpha \in E$, and that

$$M_t := Y_t - Y_0 - \int_0^t (Af)(J_s, Y_s) ds,$$

is a martingale with respect to $\{\mathcal{F}_t\}_{t \geq 0}$.

Now suppose $0 < s < t$, then using the semi-martingale property followed by the Markov property and Fubini's theorem,

$$0 \leq \mathbb{E}[Y_t \mid \mathcal{F}_s] - Y_s = \mathbb{E}_{Y_s}\left[\int_s^t (Af)(\hat{J}_u, \hat{Y}_u)du\right] = \int_0^{t-s} \mathbb{E}_{J_s,Y_s}\left[\hat{Y}_u(Af)(\hat{J}_u, 1)\right] du,$$

where $(\hat{J}, \hat{Y})$ is an independent but identically distributed copy of (J, Y). Then, applying standard calculus results, differentiating the right hand-side with respect to t and evaluating it at $t = s$ gives

$$0 \leq \mathbb{E}_{J_s,Y_s}\left[\hat{Y}_0(Af)(\hat{J}_0, 1)\right] = Y_s(Af)(J_s, 1)$$

and so $(Af)(J_s, 1) \geq 0$. Then, since J is a continuous time irreducible Markov chain, for every $\alpha \in E$ and $s > 0$, the probability of the event $\{J_s = \alpha\}$ is non-zero, thus $A^{(\alpha)} \geq 0$.

The proof of the super-martingale case is similar. □

The comparison of European and Asian call options from [26, Chapter 5, Proposition 3.1] can now be extended to exponential MAP models. For the sub-martingale case, the proof is then identical to [26], whilst the super-martingale case requires an adaptation.

Theorem 6.3.2 (Comparison of the Prices of European and Asian Call Options)
Suppose that the price process of the asset underlying an Asian call option in the equivalent martingale measure is given by $\{Y_t : t \geq 0\}$. Then, the following relations hold:

(i) if $A^{(\alpha)} > 0$ for all $\alpha \in E$, then the Asian call option, C_H^A, is cheaper than the corresponding European call option, C_H, for any strike k;

(ii) if $A^{(\alpha)} < 0$ for all $\alpha \in E$, then there exists a $K > 0$ such that the European call option, C_H, is cheaper than the Asian call option, C_H^A, for all strikes $k \leq K$.

Proof
To show (i) suppose that $A^{(\alpha)} > 0$ for all $\alpha \in E$, so that, by Proposition 6.3.1, Y is a sub-martingale. Then, the argument in the proof of [26, Chapter 5, Proposition 3.1(i)] can be followed exactly.

In the case (ii), the proof of [26, Chapter 5, Proposition 3.2(ii)] is followed, but some of the explicit calculations for Brownian motion are replaced with an inequality. In this case, by Proposition 6.3.1, Y is a super-martingale, hence it must be integrable. Moreover, by the super-martingale property, $\mathbb{E}[Y_s] > \mathbb{E}[Y_T]$ for all $0 < s < T$, thus,

$$\frac{1}{T}\int_0^T \mathbb{E}[Y_s]ds > \frac{1}{T}\int_0^T \mathbb{E}[Y_T]ds = \mathbb{E}[Y_T].$$

However, $\mathbb{E}\left[(Y_T - k)^+ + k\right] = \mathbb{E}[Y_T]$ when $k = 0$. Then, since $Y_T > 0$ for all $T \geq 0$, there exists $K \in \mathbb{R}^+$, such that for all $k \leq K$

$$\mathbb{E}[(Y_T - k)^+ + k] \leq \frac{1}{T}\int_0^T \mathbb{E}[Y_s]ds.$$

So for all $k \leq K$,

$$k \leq \frac{1}{T}\int_0^T \mathbb{E}[Y_s]ds - \mathbb{E}[(Y_T - k)^+]. \tag{6.3.4}$$

Then, for $k \leq K$, by the convexity of $(\cdot)^+$, Jensen's inequality gives

$$\mathbb{E}\left[\left(\frac{1}{T}\int_0^T Y_s ds - k\right)^+\right] \geq \mathbb{E}\left[\frac{1}{T}\int_0^T Y_s ds - k\right]^+ = \left(\mathbb{E}\left[\frac{1}{T}\int_0^T Y_s ds\right] - k\right)^+.$$

However, since $(\cdot)^+$ is monotonic, substituting (6.3.4) gives

$$\begin{aligned}\mathbb{E}\left[\left(\frac{1}{T}\int_0^T Y_s ds - k\right)^+\right] &> \left(\mathbb{E}\left[\frac{1}{T}\int_0^T Y_s ds\right] - \mathbb{E}\left[\frac{1}{T}\int_0^T Y_s ds\right] + \mathbb{E}\left[(Y_T - k)^+\right]\right)^+ \\ &= \mathbb{E}\left[(Y_T - k)^+\right].\end{aligned}$$

Hence, the European option is cheaper than the corresponding Asian option. $\square$

Chapter 7

Conclusions and Future Direction

7.1 Conclusions

Within this thesis it has been shown that many of the approaches to studying the exponential functional of a Lévy process can be extended to MAPs.

In the Lamperti-Kiu case, where $|E| = 2$, it was shown, in Theorem 3.2.1, that it is possible to find exact analogues of the finiteness result for the exponential functional of a Lévy process. Moreover, in Chapter 5, it was shown that the rate of decay of the tails can be found under Cramér's condition and in the strong subexponential case. However, the coefficient of the leading order term of the asymptotic expansion in Cramér's case was only determined in a specific subset of cases.

It was in determining the Mellin transform of the density of the exponential functional, in Chapter 4, that the greatest number of difficulties were encountered. In certain cases, it was possible to determine a matrix valued generalised Gamma function. However, strong conditions were needed to ensure sufficiently fast decay of the off-diagonal terms for this to be possible. The verification result of Theorem 4.1.2 was stated more generally and should be useful if the generalised Gamma function can be defined for a wider range of processes. However, it was possible in some cases to find the Mellin transform and use it to study the tails of the density of the exponential functional.

The European option pricing results for exponential Lévy models extend quite naturally to exponential MAP models. In Chapter 6, details were given of the Mellin transform method and the PIDE method. It was also possible to obtain Mellin transform results for the prices of Asian options. These are particularly useful in models that meet the conditions of Chapter 4 and hence allow the generalised Gamma function to be exploited. We were also able to compare the prices of European and Asian options in certain cases of interest.

7.2 Future Direction

Several of the results within this thesis were stated in the case $|E| = 2$, however, I believe they should be easily extendable to any finite E. In the case $|E| = 2$, we know that after each change, the state of J must be the opposite to it's previous state. This contrasts with the case $|E| > 2$, where we do not know which state J will be in next. However, by assuming J is irreducible, we do know that it will visit each state at some time and that it will keep returning to its initial state. This should be sufficient for the results to be extended to the case $|E|$ is finite.

I also believe it may be possible to further extend the generalised matrix Gamma function of Theorem 4.1.1 to matrices with a slower rate of decay in their off-diagonal entries. One approach would be to work with the diagonalisation of the matrices, so that we are not forced to make them diagonal artificially via the assumptions. This would manifest itself by replacing the diagonal matrix at the centre of the generalised Gamma function with a matrix raised to some power.

The financial applications of exponential MAP models could also be further extended. It would be interesting to understand the hedging portfolios implied by the pricing results. The model calibration problem would also be worth studying, particularly the question of how well the exponential MAP models may be fitted to real market prices, and within which sub-classes this can be done successfully and efficiently. This is of particular interest since we have considered both European and Asian options. It is common to calibrate a model against the more liquid European options and then use this model to value the less liquid Asian options.

Appendix A

Results From Complex Analysis

A.1 Gamma Function

The *Gamma function*, denoted Γ, is defined by

$$\Gamma(x) \coloneqq \int_0^\infty s^{x-1} e^{-s} ds,$$

for all $x \in \mathbb{C} \setminus \{-1, -2, \cdots\}$ and is undefined at the non-positive integers.

An alternative expression for the Gamma function is given by the *Weierstrass product* definition:

$$\Gamma(x) \coloneqq \frac{e^{-\gamma x}}{x} \prod_{n=1}^{\infty} \left(1 + \frac{x}{n}\right)^{-1} e^{x/n} = \lim_{N \to \infty} N^z \prod_{n=1}^{N} \left(1 + \frac{z}{n}\right)^{-1}, \tag{A.1.1}$$

where $\gamma \coloneqq \lim_{n\to\infty} - \log(n) + \sum_{k=1}^{n} k^{-1}$ is the *Euler-Mascheroni* constant. Whilst these two product expressions are equivalent, the first has the advantage that the product converges in isolation, whilst the second requires the multiplication by N^z to obtain convergence.

The Gamma function, Γ, is the unique log-convex solution to the functional equation

$$\begin{aligned} \Gamma(0) &= 1, \\ \Gamma(z+1) &= z\Gamma(z), \qquad \text{for } z \in \mathbb{C} \setminus (-\infty, 0]. \end{aligned}$$

The incomplete Gamma functions are also used within this thesis. The *lower incomplete Gamma function*, $\gamma(\cdot, \cdot)$, and *upper incomplete Gamma function*, $\Gamma(\cdot, \cdot)$, are defined by

$$\gamma(x,t) \coloneqq \int_0^t s^{x-1}e^{-s}\,ds \qquad \text{and} \qquad \Gamma(x,t) \coloneqq \int_t^\infty s^{x-1}e^{-s}\,ds,$$

for all $t > 0$ and $x \in \mathbb{C} \setminus \mathbb{Z}^-$.

The following lemma gives bounds on the decay of the Gamma function in the imaginary direction. Whilst the result isn't new, its proof is given here for completeness.

Lemma A.1.1

For all $0 < a < b < \infty$, there exist constants $C_1, C_2, T > 0$, such that for all $x + iy \in \mathcal{S}_{a,b}$ with $|y| > T$,

$$C_1 \exp\left(-\frac{1}{2}\pi|y|\right) \leq |\Gamma(x+iy)| \leq C_2 \exp\left(-\frac{1}{2}\pi|y|\right).$$

Proof

Upper Bound

Suppose $y \in \mathbb{R}$ and $x = n + z$, where $n \in \mathbb{N}_0$ and $z \in (0,1)$. Then, from the Weierstrass product definition,

$$\begin{aligned}
|\Gamma(x+iy)\Gamma(x-iy)| &= \lim_{N\to\infty} N^{2x} \prod_{k=1}^{N} \left(\frac{k}{k+x+iy}\right)\left(\frac{k}{k+x-iy}\right) \\
&= \lim_{N\to\infty} N^{2x} \prod_{k=1}^{N} \left(\frac{k^2}{(k+x)^2}\right)\left(\frac{1}{1+\frac{y^2}{(k+x)^2}}\right) \\
&= \Gamma(x)^2 \prod_{k=1}^{\infty} \frac{1}{1+\frac{y^2}{(k+x)^2}}.
\end{aligned}$$

Hence, the following inequalities can be obtained

$$\begin{aligned}
|\Gamma(x+iy)|^2 &\leq |\Gamma(x)|^2 \prod_{k=1}^{\infty} \frac{1}{1+\frac{y^2}{(n+1+k)^2}} \\
&= |\Gamma(x)|^2 \left(\prod_{k=1}^{\infty} \frac{1}{1+\frac{y^2}{(1+k)^2}}\right)\left(\prod_{k=1}^{n-1}\left(1+\frac{y^2}{(1+k)^2}\right)\right) \\
&= \left(\frac{|\Gamma(x)|}{|\Gamma(1)|}\right)^2 |\Gamma(1+yi)|^2 \left(\prod_{k=1}^{n-1}\left(1+\frac{y^2}{(1+k)^2}\right)\right) \\
&\leq |\Gamma(x)|^2 \frac{\pi|1+yi|}{|y\sinh(\pi y)|}|1+yi|^{2n}.
\end{aligned}$$

However, over any strip $\mathcal{S}_{a,b} \subset \mathbb{C}^+$ with $a > 0$ and $b < \infty$ there exists $C, T > 0$ such that, for all $|y| > T$,

$$\frac{|1+iy|^{2n+1}}{|y\sinh(\pi y)|} \leq C\exp(-\pi|y|).$$

Thus, for all $a < x < b$ and $|y| > T$,

$$|\Gamma(x+iy)| \le C\Gamma(b)\exp\left(-\frac{1}{2}\pi|y|\right).$$

Lower Bound

Suppose $y \in \mathbb{R}$ and $x = n + z$ where $n \in \mathbb{N}_0$ and $z \in (0,1)$. Then, by standard results of the Gamma function,

$$\begin{aligned}
|\Gamma(x+iy)|^2 &= |\Gamma(x)|^2 \prod_{k=1}^{\infty} \frac{1}{1+\frac{y^2}{(n+z+k)^2}} \\
&\ge |\Gamma(x)|^2 \prod_{k=1}^{\infty} \frac{1}{1+\frac{y^2}{(n+k)^2}} \\
&= |\Gamma(x)|^2 \left(\prod_{k=1}^{\infty} \frac{1}{1+\frac{y^2}{(1+k)^2}}\right)\left(\prod_{k=0}^{n-2}\left(1+\frac{y^2}{(1+k)^2}\right)\right) \\
&= \left(\frac{|\Gamma(x)|}{|\Gamma(1)|}\right)^2 |\Gamma(1+yi)|^2 \left(\prod_{k=1}^{n-2}\left(1+\frac{y^2}{(1+k)^2}\right)\right) \\
&\ge |\Gamma(x)|^2 \frac{\pi|1+yi|}{|y\sinh(\pi y)|}.
\end{aligned}$$

However, over any strip $\mathcal{S}_{a,b} \subset \mathbb{C}^+$ with $a > 0$ and $b < \infty$ there exists $C, T > 0$ such that, for all $|y| > T$,

$$\frac{|1+iy|^{2n+1}}{|y\sinh(\pi y)|} \ge C\exp(-\pi|y|).$$

Thus, for all $a < x < b$ and $|y| > T$,

$$|\Gamma(x+iy)| \ge C\Gamma(b)\exp\left(-\frac{1}{2}\pi|y|\right).$$

□

The following lemma gives bounds on the sequence from the Weierstrass definition (A.1.1).

Lemma A.1.2

For $0 < a < b$, there exist $C_1, C_2 > 0$ such that, for all $N \in \mathbb{N}$ and $s \in \mathcal{S}_{-b,-a}$,

$$C_1 < N^s \prod_{n=1}^{N}\left(1-\frac{s}{n}\right) < \frac{C_2}{|s\Gamma(-s)|}.$$

Proof

Let $N \in \mathbb{N}$ and $s \in \mathcal{S}_{-b,-a} \subset \mathbb{C}^-$, then

$$\begin{aligned}
N^s \prod_{n=1}^{N} \left(1 - \frac{s}{n}\right) &= \exp\left(s\left(\log(N) - \sum_{n=1}^{N} \frac{1}{n}\right)\right) \prod_{n=1}^{N} \left(1 - \frac{s}{n}\right) e^{s/n} \\
&= \exp\left(-\gamma s\right) \prod_{n=1}^{\infty} \left(1 - \frac{s}{n}\right) e^{s/n} \\
&\quad \times \exp\left(s\left(\gamma - \left(-\log(N) + \sum_{n=1}^{N} \frac{1}{n}\right)\right)\right) \prod_{n=N+1}^{\infty} \left(1 - \frac{s}{n}\right)^{-1} e^{-s/n} \\
&= \frac{1}{s\Gamma(-s)} \exp\left(s\left(\gamma - \left(-\log(N) + \sum_{n=1}^{N} \frac{1}{n}\right)\right)\right) \prod_{n=N+1}^{\infty} \left(1 - \frac{s}{n}\right)^{-1} e^{-s/n}.
\end{aligned}$$

However, for $s \in \mathcal{S}_{-b-a}$, it follows that $\left|1 - \frac{s}{n}\right| \geq 1 - \frac{\Re(s)}{n}$ and $\left|e^{-s/n}\right| = e^{-\Re(s)/n}$. Thus,

$$\left|\prod_{n=N+1}^{\infty} \left(1 - \frac{s}{n}\right)^{-1} e^{-s/n}\right| \leq \prod_{n=N+1}^{\infty} \left(1 - \frac{\Re(s)}{n}\right) e^{-\Re(s)/n},$$

where the right hand side converges uniformly on the bounded set $[-b, -a]$, and so is bounded. That is, there exists $C_1 > 0$, such that for all $N \in \mathbb{N}$ and $s \in \mathcal{S}_{-b,-a}$,

$$\left|\prod_{n=N+1}^{\infty} \left(1 - \frac{s}{n}\right)^{-1} e^{-s/n}\right| \leq C_1.$$

Since $-\log(N) + \sum_{n=1}^{N} \frac{1}{n}$ converges as $N \to \infty$, it is also bounded by some constant C_2. Thus,

$$\left|N^s \prod_{n=1}^{N} \left(1 - \frac{s}{n}\right)\right| \leq \frac{1}{|s\Gamma(-s)|} \exp(\Re(s)(\gamma + C_2)) C_1 \leq C_3 \frac{1}{|s\Gamma(-s)|},$$

for some constant $C_3 > 0$.

Now consider the reciprocal. This can be written as,

$$\left|N^{-s} \prod_{n=1}^{N} \left(1 - \frac{s}{n}\right)^{-1}\right| = \left|\exp\left(-s\left(\log(N) - \sum_{n=1}^{N} \frac{1}{n}\right)\right) \prod_{n=1}^{N} \left(1 - \frac{s}{n}\right)^{-1} e^{-s/n}\right|,$$

then using the inequalities from above,

$$\left|\prod_{n=1}^{N} \left(1 - \frac{s}{n}\right)^{-1} e^{-s/n}\right| \leq \prod_{n=1}^{N} \left(1 - \frac{\Re(s)}{n}\right)^{-1} e^{-\Re(s)/n},$$

which converges uniformly on the bounded set $[-b, -a]$. That is, there exists $C_4 > 0$ such that for all $N \in \mathbb{N}$ and $s \in \mathcal{S}_{-b,-a}$

$$\left|\prod_{n=1}^{N} \left(1 - \frac{s}{n}\right)^{-1} e^{-s/n}\right| \leq C_4.$$

Thus, using the bound on $\log(N) - \sum_{n=1}^{N} \frac{1}{n}$,

$$\left| N^{-s} \prod_{n=1}^{N} \left(1 - \frac{s}{n}\right)^{-1} \right| \leq C_5,$$

for some constant $C_5 > 0$. □

A.2 Other Special Functions

Modified Bessel Function of the First Kind

The *modified Bessel function of the first kind* with parameter 1 is denoted by I_1 and has the series representation

$$I_1(x) := \sum_{n=0}^{\infty} \frac{1}{n!(n+1)!} \left(\frac{x}{2}\right)^{2n+1}.$$

It is related to the *standard Bessel function of the first kind*, J_α, by the relation $I_1(x) = i^{-1} J_1(ix)$.

Generalised Hypergeometric Function

The *generalised Hypergeometric function* is given by the series

$${}_pF_q(a_1, \cdots, a_p; b_1, \cdots, b_q; x) := \sum_{n=0}^{\infty} \frac{(a_1)_n \cdots (a_p)_n x^n}{(b_1)_n, \cdots (b_q)_n n!},$$

for $z \in \mathbb{C}$, $p, q \in \mathbb{N}$, where $(a)_n$ denotes the Pochhammer symbol and is given by

$$(a)_n := \begin{cases} 1 & \text{if } n = 0; \\ a(a+1)\cdots(a+n-1) & \text{if } n \geq 1. \end{cases}$$

A.3 Integral Transforms

Several integral transforms are used throughout this thesis. A brief overview of their definitions is given below.

Fourier Transform

For a function $f : \mathbb{R} \to \mathbb{R}$ such that $f \in L^1(\mathbb{R})$, it's Fourier transform is given by

$$\hat{f}(u) := \{\mathcal{F}f\}(u) := \int_{\mathbb{R}} e^{-iux} f(x) dx.$$

If $\hat{f} \in L^1(\mathbb{R})$, then the inverse Fourier transform is given by

$$f(x) := \left\{\mathcal{F}^{-1}\hat{f}\right\}(x) := \frac{1}{2\pi}\int_{\mathbb{R}} e^{ixu}\hat{f}(u)du.$$

For further details the reader is referred to [50].

Mellin Transform

The main reference considered for the Mellin transform is [43]. The *Mellin transform* of a function $f : \mathbb{R}^+ \to \mathbb{R}$ is given by

$$\{\mathcal{M}f\}(u) := \int_0^\infty x^{u-1}f(x)dx,$$

whenever such an integral converges. Moreover, if there is a strip $\mathcal{S}_{a,b} := \{s \in \mathbb{C} : a < \Re(s) < b\}$ in which $\{\mathcal{M}f\}(s)$ is analytic and tends to zero uniformly as $|\Im(s)| \to \infty$, then the *inverse Mellin transform* is given by

$$\left\{\mathcal{M}^{-1}f\right\}(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} x^{-s}f(s)ds,$$

for any $c \in (a, b)$, such that the integral converges absolutely. Then, the *Mellin inversion theorem* states that $f = \mathcal{M}^{-1}\mathcal{M}f$.

Laplace Transform

The *Laplace transform* of a function $f : \mathbb{R}^+ \to \mathbb{R}$ is given by

$$\{\mathcal{L}f\}(u) := \int_0^\infty e^{-sx}f(x)dx,$$

whenever such an integral converges. The *inverse Laplace transform* of f is given by

$$\left\{\mathcal{L}^{-1}f\right\}(x) := \frac{1}{2\pi i}\lim_{T\to\infty}\int_{\gamma-iT}^{\gamma+iT} e^{xs}f(s)ds$$

where $\gamma \in \mathbb{R}$ is to the right of all singularities of f. For further details on the Laplace transform see [33].

Multiplicative Convolution

For two functions $f, g : \mathbb{R}^+ \to \mathbb{R}$, the *multiplicative convolution*, also know as the *Mellin type convolution*, is defined to be

$$\{f * g\}(x) = \int_0^\infty f(y)g\left(\frac{x}{y}\right)\frac{1}{y}dy,$$

for $x \in \mathbb{R}^+$, whenever such an integral converges.

The multiplicative convolution is more natural than the standard convolution when working in the context of a Mellin transform. For example, if the Mellin transforms of f and g exist, then

$$\{\mathcal{M}(f * g)\}(x) = \{\mathcal{M}f\}(x)\{\mathcal{M}g\}(x).$$

Appendix B

Calculations of Finance Applications

This appendix includes the details of some of the calculations of the examples from Chapter 6, which were too lengthy to include in the main text of the thesis.

B.1 Mellin Inversion

In Example 6.1.1, the following Mellin inversion was needed to compute the price of a call option:

$$C_\alpha(k) = \frac{1}{2\pi i}\int_{c+1+i\mathbb{R}} \frac{k^{-s+1}}{(s-1)s}\frac{2e^{-2qt}G_\alpha e^{qt(G_+ + G_-)}}{(G_+ + G_-)}ds + \frac{1}{2\pi i}\int_{c+1+i\mathbb{R}} \frac{k^{-s+1}e^{-2qt}}{s(s-1)}\left(\frac{G_{-\alpha} - G_\alpha}{G_+ + G_-}\right)ds,$$

for $c \in (0, \min(\lambda_+, \lambda_-))$, $\alpha \in E$ and $k \in \mathbb{R}^+$. As an intermediary step, for $c \in (1, \min(\lambda_+, \lambda_-))$, consider

$$D_\alpha^c(k) \coloneqq \frac{1}{2\pi i}\int_{c+1+i\mathbb{R}} \frac{k^{-(s-1)}}{(s-1)s}\frac{e^{-2qt}}{(G_+ + G_-)}G_\alpha ds = \frac{1}{2\pi i}\int_{c+i\mathbb{R}} \frac{k^{-(s-1)}}{(s-1)s}e^{-2qt}\frac{\lambda_\alpha(\lambda_{-\alpha} - s)}{2\lambda_+\lambda_- - s(\lambda_+ + \lambda_-)}ds,$$

where the substitution $G_\alpha(z) = \lambda_\alpha/(\lambda_\alpha - z)$, for $z < \lambda_\alpha$ and $\alpha \in E$, has been made. The poles of the integrand are at 0, 1 and $s^* \coloneqq 2\lambda_+\lambda_-/(\lambda_+ + \lambda_-)$. For $c \in (1, \min(\lambda_+, \lambda_-))$, it follows that $0, 1 < c$ and $s^* > c$ and the residues of the integrand are:

$$\mathrm{Res}(1) = \frac{k^{-(1-1)}}{1} e^{-2qt} \frac{\lambda_\alpha(\lambda_{-\alpha} - 1)}{2\lambda_+\lambda_- - 1(\lambda_+ + \lambda_-)} = e^{-2qt} \frac{\lambda_\alpha(\lambda_{-\alpha} - 1)}{2\lambda_+\lambda_- - (\lambda_+ + \lambda_-)},$$
$$\mathrm{Res}(0) = \frac{k^{-(0-1)}}{(0-1)} e^{-2qt} \frac{\lambda_\alpha(\lambda_{-\alpha} - 0)}{2\lambda_+\lambda_- - 0(\lambda_+ + \lambda_-)} = -\frac{1}{2} k e^{-2qt},$$
$$\mathrm{Res}(s^*) = \frac{k^{-(s^*-1)} e^{-2qt} (\lambda_\alpha - \lambda_{-\alpha})}{2\left(2\lambda_+\lambda_- - (\lambda_+ + \lambda_-)\right)}.$$

It is clear that $\lim_{c\to\infty} D^c_\alpha(k) = 0$ when $k \geq 1$ and $\lim_{c\to-\infty} D^c_\alpha(k) = 0$ when $k < 1$, thus, by Cauchy's residue theorem,

$$D_\alpha(k) = \begin{cases} -\frac{1}{2} k e^{-2qt} + e^{-2qt} \frac{\lambda_i(\lambda_{-i} - 1)}{2\lambda_+\lambda_- - (\lambda_+ + \lambda_-)}, & \text{if } k < 1; \\ \frac{k^{-(s^*-1)} e^{-2qt} (\lambda_\alpha - \lambda_{-\alpha})}{2(2\lambda_+\lambda_- - (\lambda_+ + \lambda_-))}, & \text{if } k \geq 1. \end{cases} \tag{B.1.1}$$

Moreover, from a direct calculation we see that if $k > 1$,

$$D_{-\alpha}(k) - D_\alpha(k) = \begin{cases} \frac{e^{-2qt}(\lambda_{-\alpha} - \lambda_\alpha)}{2\lambda_+\lambda_- - (\lambda_+ + \lambda_-)}, & \text{if } k < 1; \\ \frac{k^{-s^*+1} e^{-2qt}(\lambda_{-\alpha} - \lambda_\alpha)}{2\lambda_+\lambda_- - (\lambda_+ + \lambda_-)}, & \text{if } k \geq 1. \end{cases} \tag{B.1.2}$$

Now consider the terms involving $e^{qt(G_+ + G_-)}$. Let I_1 be the *modified Bessel function* of the first kind of order 1 (see Appendix A.2 for further details). Then, for $c \in \mathbb{R}$ and $x < 0$, from the series expansion of I_1, it follows that

$$\mathcal{M}_k \left\{ \mathbb{1}_{\{k\geq 1\}} \frac{I_1\left(c\sqrt{\log(k)}\right)}{\sqrt{\log(k)}} \right\}(x) = \sum_{m=0}^{\infty} \frac{c^{2m+1}}{m!(m+1)!2^{2m+1}} \int_1^\infty \frac{k^{x-1} \log(k)^{m+\frac{1}{2}}}{\log(k)^{\frac{1}{2}}} dk.$$

Then, consider the change of variables $y = \log(k)$ to obtain

$$\int_1^\infty k^{x-1} \log(k)^m dk = \int_0^\infty e^{yx} y^m dy = \frac{m!}{(-x)^{m+1}}.$$

Hence, for $x < 0$,

$$\mathcal{M}_k \left\{ \mathbb{1}_{\{k\geq 1\}} \frac{I_1\left(c\sqrt{\log(k)}\right)}{\sqrt{\log(k)}} \right\}(x) = \sum_{m=0}^{\infty} \frac{c^{2m+1}}{m!(m+1)!2^{2m+1}} \frac{m!}{(-x)^{m+1}} = \frac{2}{c}\left(\exp\left(-\frac{c^2}{4x}\right) - 1\right).$$

Now set $c \coloneqq 2\sqrt{qt\lambda_\alpha}$ and consider

$$R_\alpha(k) \coloneqq \delta_1(k) + \sqrt{qt\lambda_\alpha}\, \mathbb{1}_{\{k\geq 1\}} k^{-\lambda_\alpha} \frac{I_1\left(2\sqrt{qt\lambda_\alpha \log(k)}\right)}{\sqrt{\log(k)}},$$

where δ_k denotes the Dirac delta distribution. Then, by the shift rule for the Mellin transform, for $x < \lambda_\alpha$,

$$\{\mathcal{M} R_\alpha\}(x) = \exp\left(\frac{qt\lambda_\alpha}{\lambda_\alpha - x}\right).$$

Moreover, notice that $e^{-qt}R_\alpha(e^x)$ is the probability density of a compound Poisson process of rate q, with exponential jumps of rate λ_α, at time t. Hence, $e^{-qt}R_\alpha(e^x)$ is the density of ξ_t^α for all $t>0$.

By letting $*$ denote the *Mellin type convolution*, it follows that

$$\mathcal{M}\left\{D_\alpha * R_+ * R_-\right\}(s) = \frac{ke^{-2qt}}{s(s-1)}\frac{G_\alpha}{(G_+ + G_-)}\exp\left(\frac{qt\lambda_+}{\lambda_+ - x} + \frac{qt\lambda_-}{\lambda_- - x}\right).$$

Hence, the Mellin transform of C_α is given by

$$\left\{\mathcal{M}_k C_\alpha(k)\right\}(s) = 2\mathcal{M}\left\{D_\alpha * R_+ * R_-\right\}(s) + \{\mathcal{M}D_{-\alpha}\}(s) - \{\mathcal{M}D_\alpha\}(s),$$

that is,

$$C_\alpha(k) = 2\{D_\alpha * R_+ * R_-\}(k) + D_{-\alpha}(k) - D_\alpha(k), \tag{B.1.3}$$

where an explicit expression for $D_{-\alpha}(k) - D_\alpha(k)$ is given in (B.1.2).

B.2 Series Expansion

To continue with Example 6.1.1 it useful to express the triple convolution (6.1.14) as a triple infinite series via the following calculations.

For $\alpha \in E$, define the function

$$J_\alpha(k) \coloneqq \sqrt{qt\lambda_\alpha}k^{-\lambda_\alpha}\frac{I_1\left(2\sqrt{qt\lambda_\alpha\log(k)}\right)}{\sqrt{\log(k)}}, \tag{B.2.1}$$

so that $R_\alpha = \delta_1(k) + \mathbb{1}_{\{k\geq 1\}}J_\alpha(k)$. Then, first consider the multiplicative convolution $R_+ * R_-$, which is given by,

$$\begin{aligned}(R_+ * R_-)(k) &= \int_0^\infty R_+(x)R_-\left(\frac{k}{x}\right)\frac{1}{x}dx \\ &= \delta_1(k) + \mathbb{1}_{\{k\geq 1\}}J_-(k) + \mathbb{1}_{\{k\geq 1\}}\frac{1}{k}J_+(k) + \int_1^k J_+(x)J_-\left(\frac{k}{x}\right)\frac{1}{x}dx.\end{aligned}$$

Now consider the final integral in more detail. Making use of the definition of J_α from (B.2.1), for $k>1$,

$$\begin{aligned}\int_1^k J_+(x)J_-\left(\frac{k}{x}\right)\frac{1}{x}dx &= qt\sqrt{\lambda_+\lambda_-}\int_1^k \frac{x^{-\lambda_+}I_1(2\sqrt{qt\log(x)})}{\sqrt{\log(x)}}\left(\frac{k}{x}\right)^{-\lambda_-}\frac{I_1(2\sqrt{qt\log(k/x)})}{\sqrt{\log(k/x)}}\frac{1}{x}dx \\ &= qt\sqrt{\lambda_+\lambda_-}k^{-\lambda_-}\int_0^{\log(k)} e^{(\lambda_- - \lambda_+)z}\frac{I_1\left(2\sqrt{qtz}\right)I_1\left(2\sqrt{qt}\sqrt{\log(k)-z}\right)}{\sqrt{\log(k)-z}}dz,\end{aligned}$$

where the second equality is obtained via the substitution $z = \log(x)$. From the series expansions of the modified Bessel function I_1 and the exponential, it follows that

$$\int_1^k J_+(x)J_-\left(\frac{k}{x}\right)\frac{1}{x}dx$$
$$= qt\sqrt{\lambda_+\lambda_-}k^{-\lambda_-}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{r=0}^{\infty}\frac{(qt)^{m+n+1}(\lambda_- - \lambda_+)^r}{m!(m+1)!n!(n+1)!r!}\int_0^{\log(k)} z^{m+r}\left(\log(k)-z\right)^n dz.$$

Then, by using integration by parts for $n > 0$,

$$\int_0^{\log(k)} z^{m+r}\left(\log(k)-z\right)^n dz = \frac{n!(r+m)!}{(r+m+n+1)!}\log(k)^{r+m+n+1},$$

and so,

$$\int_1^k J_+(x)J_-\left(\frac{k}{x}\right)\frac{1}{x}dx = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{r=0}^{\infty}a_{n,m,r}(q,t,\lambda_-,\lambda_+)k^{-\lambda_-}\log(k)^{r+m+n+1},$$

where

$$a_{n,m,r}(q,t,\lambda_-,\lambda_+) := \frac{(qt)^{m+n+2}\sqrt{\lambda_+\lambda_-}(\lambda_- - \lambda_+)^r(r+m)!}{m!(m+1)!(n+1)!r!(r+m+n+1)!}.$$

Now consider the convolution of this series with D_α. From (B.1.1) there exist constants $d_1^\alpha, d_2^\alpha, d_3^\alpha$ and c_α, such that

$$D_\alpha(k) = \begin{cases} d_1^\alpha k + d_2^\alpha & \text{if } k < 1; \\ d_3^\alpha k^{c_\alpha} & \text{if } k \geq 1. \end{cases}$$

Hence,

$$\left(D_\alpha * J_+ * J_-\right)(k) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{r=0}^{\infty}a_{n,m,r}(q,t,\lambda_-,\lambda_+)$$
$$\times\left(\int_0^1\left(\frac{k}{x}\right)^{-\lambda_-}\log\left(\frac{k}{x}\right)^{r+m+n+1}\left(d_1^\alpha x + d_2^\alpha\right)\frac{1}{x}dx + \int_1^\infty\left(\frac{k}{x}\right)^{-\lambda_-}\log\left(\frac{k}{x}\right)^{r+m+n+1} d_3^\alpha x^{c_\alpha}\frac{1}{x}dx\right).$$

Now consider the following integrals, for $l \in (0,\infty)$, and make the substitution $t = (a+1)\log(k/x)$ to obtain

$$\int_0^l x^a \log\left(\frac{k}{x}\right)^N dx = \frac{k^{a+1}}{(a+1)^{N+1}}\int_{(a+1)\log(k/l)}^{\infty} e^{-t}t^N dt = \frac{k^{a+1}}{(a+1)^{N+1}}\Gamma\left(N+1,(a+1)\log\left(\frac{k}{l}\right)\right),$$
$$\int_l^\infty x^a \log\left(\frac{k}{x}\right)^N dx = \frac{k^{a+1}}{(a+1)^{N+1}}\int_0^{(a+1)\log(k/l)} e^{-t}t^N dt = \frac{k^{a+1}}{(a+1)^{N+1}}\gamma\left(N+1,(a+1)\log\left(\frac{k}{l}\right)\right),$$

where $\Gamma(\cdot,\cdot)$ and $\gamma(\cdot,\cdot)$ are the *upper* and *lower incomplete Gamma functions*, respectively. Thus, it follows that

$$\left(D_\alpha * J_+ * J_-\right)(k) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{r=0}^{\infty}\left(g_1(n,m,r,\lambda_-,k) + g_2(n,m,r,\lambda_-,k) + g_3(n,m,r,\lambda_-,k)\right),$$

where,

$$
\begin{aligned}
g_1(n,m,r,\lambda_-,k) &:= a_{n,m,r}(q,t,\lambda_+,\lambda_-)\frac{d_1^\alpha k}{(\lambda_-+1)^{r+m+n+2}}\Gamma(r+m+n+2,(\lambda_-+1)\log(k)),\\
g_2(n,m,r,\lambda_-,k) &:= a_{n,m,r}(q,t,\lambda_+,\lambda_-)\frac{d_2^\alpha}{\lambda_-^{r+m+n+2}}\Gamma(r+m+n+2,\lambda_-\log(k)),\\
g_3(n,m,r,\lambda_-,k) &:= a_{n,m,r}(q,t,\lambda_+,\lambda_-)\frac{d_3^\alpha k^{c_\alpha}}{(c_\alpha+\lambda_-)^{r+m+n+2}}\gamma(r+m+n+2,(c_\alpha+\lambda_-)\log(k)).
\end{aligned}
$$

Now compute the multiplicative convolution

$$
\begin{aligned}
\left\{\left\{\mathbb{1}_{\{x\geq 1\}}\frac{1}{x}J_\beta(x)\right\}*D_\alpha(x)\right\}(k) &= \int_0^\infty D_\alpha(x)\mathbb{1}_{\{k\geq x\}}\frac{x}{k}J_\beta\left(\frac{k}{x}\right)\frac{1}{x}dx\\
&= \frac{1}{k}\int_0^{k\wedge 1}\left(d_1^\alpha x+d_2^\alpha\right)J_+\left(\frac{k}{x}\right)dx+\frac{1}{k}\int_{k\wedge 1}^k d_3^\alpha x^{c_\alpha}J_\beta\left(\frac{k}{x}\right)dx.
\end{aligned}
$$

Consider the general integral

$$
\begin{aligned}
\int_l^u x^a J_\beta\left(\frac{k}{x}\right)dx &= \sqrt{qt\lambda_\beta}k^{-\lambda_\beta}\int_l^u x^{a+\lambda_\beta}\frac{I_1\left(2\sqrt{qt\lambda_\beta\log(k/x)}\right)}{\sqrt{\log(k/x)}}dx\\
&= qt\lambda_\beta\sum_{m=0}^\infty\frac{k^{-\lambda_\beta}}{m!(m+1)!}\int_l^u x^{\lambda_\beta+a}\log\left(\frac{k}{x}\right)^m dx.
\end{aligned}
$$

Then, using the previous results to evaluate the integral,

$$
\begin{aligned}
\int_l^u x^a J_\beta\left(\frac{k}{x}\right) &= qt\lambda_\beta\sum_{m=0}^\infty\frac{k^{-\lambda_\beta}}{m!(m+1)!}\frac{k^{\lambda_\beta+a+1}}{(\lambda_\beta+a+1)^{m+1}}\\
&\times\left(\Gamma\left(m+1,(\lambda_\beta+a+1)\log\left(\frac{k}{u}\right)\right)-\Gamma\left(m+1,(\lambda_\beta+a+1)\log\left(\frac{k}{l}\right)\right)\right),
\end{aligned}
$$

so,

$$
\begin{aligned}
&\left\{\left\{\mathbb{1}_{\{x\geq 1\}}\frac{1}{x}J_\beta(x)\right\}*D_\alpha(x)\right\}(k)\\
&\quad=\sum_{m=0}^\infty\frac{qt\lambda_\beta}{m!(m+1)!}\left(F(m,d_1^\alpha,\lambda_\beta,1,k)+F(m,d_2^\alpha,\lambda_\beta,0,k)+f(m,d_3^\alpha,\lambda_\beta,c_\alpha,k)\right)
\end{aligned}
$$

and

$$
\begin{aligned}
&\left\{\left\{\mathbb{1}_{\{x\geq 1\}}J_\beta(x)\right\}*D_\alpha(x)\right\}(k)\\
&\quad=\sum_{m=0}^\infty\frac{qt\lambda_\beta}{m!(m+1)!}\left(F(m,d_1^\alpha,\lambda_\beta,2,k)+F(m,d_2^\alpha,\lambda_\beta,1,k)+f(m,d_3^\alpha,\lambda_\beta,c_\alpha+1,k)\right),
\end{aligned}
$$

where,

$$
\begin{aligned}
F(m,d,\lambda,c,k) &:= \frac{dk^c\Gamma(m+1,(\lambda+c+1)\log(k\vee 1)))}{(\lambda_\alpha+c+1)^{m+1}},\\
f(m,d,\lambda,c,k) &:= \frac{dk^c\gamma(m+1,(\lambda+c+1)\log(k\vee 1)))}{(\lambda_\alpha+c+1)^{m+1}}.
\end{aligned}
$$

Finally, notice that $\{D_\alpha * \delta_1\}(k) = D_\alpha(k)$. Thus, putting these components together gives

$$
\begin{aligned}
C_\alpha(k) = &\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{r=0}^{\infty} 2\left(g_1(n,m,r,\lambda_-,k) + g_2(n,m,r,\lambda_-,k) + g_3(n,m,r,\lambda_-,k)\right) \\
&+ \sum_{m=0}^{\infty} \frac{qt\lambda_+}{m!(m+1)!}\left(F(m,d_1^\alpha,\lambda_+,1,k) + F(m,d_2^\alpha,\lambda_+,0,k) + f(m,d_3^\alpha,\lambda_+,c_\alpha,k)\right) \\
&+ \sum_{m=0}^{\infty} \frac{qt\lambda_-}{m!(m+1)!}\left(F(m,d_1^\alpha,\lambda_-,2,k) + F(m,d_2^\alpha,\lambda_-,1,k) + f(m,d_3^\alpha,\lambda_-,c_\alpha+1,k)\right) \\
&+ \mathbb{1}_{\{k<1\}}\left(d_1^{-\alpha} + d_2^{-\alpha}\right) + \mathbb{1}_{\{k\geq 1\}}\left(d_3^{-\alpha} k^{c_{-\alpha}}\right).
\end{aligned}
$$

Appendix C

Other Useful Results

C.1 Hadamard's Inequality

Hadamard's inequality [25, pp 233, Chapter 14, Theorem 14.1.1] states that, for a matrix $A \in \mathbb{C}^{N\times N}$ with columns $a_1, \cdots, a_n$,

$$|\det(A)| \leq \prod_{n=1}^{N} \|a_i\|,$$

where $\|\cdot\|$ denotes the Euclidean norm. From this, it follows that

$$|\det(A)| \leq \max_{i=,\cdots,N} \|a_i\|^N = \|A\|^N.$$

Moreover, since $\mathbb{C}^{N\times N}$ is finite dimensional, there exists some constant $C > 0$, such that for any $A \in \mathbb{C}^{N\times N}$

$$|\det(A)| \leq C\|A\|_{l^1}^N,$$

where $\|\cdot\|_{l^1}^N$ denotes the matrix norm induced by the 1-norm on $\mathbb{C}^N$.

C.2 Periodic Functions

The following lemma is a result that is used within the proof of [34, pp 11, Proposition 2]. Here a full proof is given for completeness.

Lemma C.2.1

Suppose $w : \mathbb{C} \to \mathbb{C}$ is an entire periodic function, with period 1, and there exists $T, C > 0$

and $\alpha \in (0,2)$, such that $|w(s)| \leq C\exp(\alpha\pi|\Im(s)|)$, for all $s \in \mathbb{C}$ with $|\Im(s)| > T$. Then, w is constant.

Proof

Since w is an entire periodic function with period 1, it is given by the *Fourier series*

$$w(s) = \sum_{n=-\infty}^{\infty} a_n \exp(2\pi i n s) = a_0 + W_+(\exp(2\pi i s)) + W_-(\exp(-2\pi i s)),$$

where $\{a_n\}_{n\in\mathbb{Z}}$ are the *Fourier coefficients* and, for $z \in \mathbb{C}$,

$$W_+(z) := \sum_{n=1}^{\infty} a_n z^n \qquad \text{and} \qquad W_-(z) := \sum_{n=1}^{\infty} a_{-n} z^n.$$

Since $\lim_{\Im(s)\to-\infty} \Re(2\pi i s) = \infty$, it follows that $|\exp(2\pi i s)| \to \infty$ and $\exp(-2\pi i s) \to 0$ as $|\Im(s)| \to -\infty$. Thus, the assumption $|w(s)| < C\exp(\alpha\pi|\Im(s)|)$ for large $|\Im(s)|$ implies $|W_+(z)| < C|z|^{\alpha/2}$ for large $|z|$.

For any $R > 0$ and $z \in \mathbb{C}$, let $B_R(z) := \{x \in \mathbb{C} : |z - x| < R\}$. Then, by Cauchy's integral formula, for each $m \in \mathbb{N}$ and for sufficiently large $R \in \mathbb{R}^+$,

$$|W_+^{(m)}(z)| \leq \frac{m!}{2\pi}\int_{B_R(z)} \frac{|W_+(x)|}{|x-z|^{m+1}}dx \leq \frac{m!}{2\pi}\int_{B_R(z)} \frac{C|x|^{\alpha/2}}{|x-z|^{m+1}}dx,$$

where, by taking $z = 0$,

$$\left|W_+^{(m)}(0)\right| \leq \frac{Cm!}{2\pi}\int_{B_R(0)} \frac{R^{\alpha/2}}{R^{m+1}}dx = Cm!|R|^{\frac{\alpha}{2}-m}.$$

Since $\frac{\alpha}{2} - m < 0$, taking the limit as $R \to \infty$ gives $W_+^{(m)}(0) = 0$, hence $W_+ \equiv 0$. Similarly, $W_- \equiv 0$ and so w is constant.

□

C.3 Lévy Processes

The following lemma gives a standard result for Lévy processes. In the absence of a good reference, the proof has been included for completeness.

Lemma C.3.1
Suppose $\{Z_t : t \geq 0\}$ is a Lévy process and $\psi(z) := \log(\mathbb{E}[\exp(zZ_1)])$. Then, $\psi(iu) = \mathcal{O}(u^2)$ as $u \to \pm\infty$.

Proof

From the Lévy-Khintchine formula, there exists $a, \sigma \in \mathbb{R}$ and a measure μ on $\mathbb{R}$ such that, for all $u \in \mathbb{R}$,

$$\psi(iu) = +aiu - \frac{1}{2}\sigma^2 u^2 + \int_{\mathbb{R}} \left(e^{iux} - 1 - \mathbb{1}_{\{|x|<1\}} iux\right) \mu(dx).$$

However, considering the integral around the origin, for $u > 1$,

$$\begin{aligned}\left| \int_{|x|<\frac{1}{u}} \left(e^{iux} - 1 - iux\right) \mu(dx) \right| &\leq \sum_{n=2}^{\infty} \int_{|x|<\frac{1}{u}} \left| \frac{(iux)^n}{n!} \right| \mu(dx) \\ &\leq \sum_{n=2}^{\infty} \frac{u^2}{n!} \int_{|x|<\frac{1}{u}} x^2 \mu(dx) \leq u^2 \sum_{n=2}^{\infty} \frac{1}{n!} \int_{|x|<1} x^2 \mu(dx),\end{aligned} \tag{C.3.1}$$

which is finite by the conditions on μ of the Lévy-Khintchine formula. Moreover, the remainder of the integral can be bounded by

$$\begin{aligned}&\left| \int_{|x|>\frac{1}{u}} (e^{iux} - 1 - \mathbb{1}_{\{|x|<1\}} iux) \mu(dx) \right| \\ &\qquad \leq 2\mu\left(\frac{1}{u}, \infty\right) + 2\mu\left(-\infty, -\frac{1}{u}\right) + u\left(\mu\left(\frac{1}{u}, 1\right) + \mu\left(-1, -\frac{1}{u}\right)\right),\end{aligned}$$

where all the coefficients are finite by the assumptions on μ. Thus, $\psi(iu)$ is bounded by a quadratic in u and the result of the lemma follows.

□

The following lemma bounds the probability distribution of the supremum of a Lévy process over an exponentially distributed time interval by a function of the probability distribution of the process at the end of the interval. It is a slight variation of [54, Lemma 1], where the time interval considered is fixed.

Lemma C.3.2

Let X be a Lévy process, τ be an independent exponentially distributed random variable and suppose $0 < u_0 < u$. Then,

$$\mathbb{P}\left(\sup_{0\leq s<\tau} X_s > u\right) \leq \frac{\mathbb{P}\left(X_\tau \geq u - u_0\right)}{\mathbb{P}\left(X_\tau \geq -u_0\right)}. \tag{C.3.2}$$

Proof

Let $S_u := \inf\{t \geq 0 \ : \ X_t > u\}$. Then, by the càdlàg property $X_{S_u} \geq u$. Hence, from independent increments of X and the memoryless property τ, we have that

$$\begin{aligned}\mathbb{P}(S_u < \tau; X_\tau < u - u_0) &\leq \mathbb{P}(S_u < \tau; X_\tau - X_{S_u} < -u_0) \\ &= \mathbb{P}(S_u < \tau; \tilde{X}_{\tilde{\tau}} < -u_0) = \mathbb{P}(S_u < \tau)\mathbb{P}(X_\tau < -u_0),\end{aligned}$$

where $\tilde{X}$ and $\tilde{\tau}$ are independent and identically distributed copies of X and τ, respectively. Then, (C.3.2) can be obtained by rearranging the inequality

$$\begin{aligned}\mathbb{P}(S_u < \tau) &\leq \mathbb{P}\left(X_\tau \geq u - u_0\right) + \mathbb{P}\left(S_u < \tau;\ X_\tau < u - u_0\right) \\ &\leq \mathbb{P}\left(X_\tau \geq u - u_0\right) + \mathbb{P}(S_u < \tau)\mathbb{P}(X_\tau < -u_0).\end{aligned}$$

$\square$

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